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THE TRIAD SYSTEMS OF THIRTEEN LETTERS*

BY

F. N. COLE

Distributions of thirteen letters in triads were given by Kirkman,† Reiss,‡ Netto,§ and De Vries.‖ In his doctor dissertation Zulauf ¶ reduced the known systems to two, those of Kirkman, Reiss, and De Vries being equivalent and having an intransitive group of order 6, while that of Netto has a transitive group of order 39. Exhaustive enumerations by De Pasquale ** and Brunel †† have shown that the two distributions mentioned are the only possible ones for thirteen letters, apart from mere variations in notation.

The investigations of De Pasquale and Brunel are based on the Kantor configurations $(3, 3)_{10}$, to which the triad systems in thirteen letters reduce on the suppression of those triads which contain any one of the three letters of a selected triad. While this method leaves little to be desired in point of efficiency, it is quite possible to obtain the systems by the theory of substitutions alone, as is accomplished in the present paper.

A guiding principle here employed is that of "interlacing," the nature of which is sufficiently indicated by the instance $x\,a\,b,\ x\,c\,d;\ y\,a\,c,\ y\,b\,d$. It may be permitted to speak of this distribution as an interlacing of x and y. From elementary considerations it is plain that such interlacing is unavoidable in the triad system of seven letters, but is excluded in the case of nine letters. For thirteen letters it is again unavoidable. The proof is sufficiently tedious, but straightforward.

1. Thus, excluding interlacing and disregarding mere matters of notation, we may begin the table of triads with

* Presented to the Society, September 11,1912.

† *Theorems on combinations*, Cambridge and Dublin Mathematical Journal, vol. 8 (1853).

‡ *Ueber eine Steinersche kombinatorische Aufgabe*, Crelle's Journal, vol. 56 (1859).

§ *Zur Theorie der Tripelsysteme*, Mathematische Annalen, vol. 42 (1892).

‖ *Zur Theorie der Tripelsysteme*, Rendiconti del Circolo Matematico di Palermo, vol. 8 (1894).

¶ *Ueber Tripelsysteme von 13 Elementen*, Marburg, 1897.

** *Sui sistemi ternari di 13 elementi*, Rendiconti R. Istituto Lombardo, ser. 2, vol. 32 (1899).

†† *Sur les deux systemes de triades de treize éléments*, Journal de Mathématiques, ser. 5, vol. 7 (1901).

1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13
	2 4 6	2 5 8	2 7 10	2 9 12	2 11 13

The system thus far developed is unaltered by the dihedral group of order 20: $G = \{ (1\,2)\,(3)\,(4\,9\,11\,6\,8\,13\,7\,5\,12\,10), (1\,2)\,(3)\,(4)\,(5\,6)\,(7\,8)\,(9\,10)\,(11\,12)\,(13) \}$.

Continuing the formation of the system by taking the pair 4 3 with a third letter, we find three possibilities not equivalent under G, viz., 4 3 9, 4 3 11, 4 3 13. The table below shows the continuation of the building process. In each division any one of the rows on the right may be combined with any row on the left.

I. 4 3 9. Invariant under (4 9) (5 8) (6 12) (7 13) (10 11).

4 7 11	4 8 13	4 10 12	9 13 10	9 5 7	9 11 6
4 7 12	4 8 11	4 10 13	9 13 6	9 5 10	9 11 7
4 7 13	4 8 10	4 11 12	9 13 7	9 5 11	9 10 6
4 7 13	4 8 11	4 10 12	9 13 7	9 5 10	9 11 6

II. 4 3 11. Invariant under (1 2) (4 11) (5 13) (6 10) (8 12).

4 7 8	4 9 13	4 10 12	11 7 12	11 9 5	11 6 8
4 7 9	4 8 12	4 10 13	11 7 9	11 8 12	11 5 6
4 7 13	4 8 12	4 9 10	11 7 8	11 12 6	11 5 9
4 7 13	4 8 12	4 9 10	11 5 7	11 8 12	11 9 6

III. 4 3 13. Invariant under (4 13) (5 12) (6 11) (7 10) (8 9).

4 7 8	4 9 10	4 11 12	13 10 9	13 7 8	13 5 6
4 7 8	4 9 11	4 10 12	13 10 9	13 6 8	13 7 5
4 7 9	4 8 10	4 11 12	13 10 8	13 9 7	13 5 6
4 7 11	4 8 12	4 9 10	13 10 6	13 5 9	13 8 7

In every one of these cases there is conflict in the formation of the triads, e. g., 4 7 13 and 9 13 7, or interlacing, or these conditions arise as soon as one attempts to extend the formation further.

2. Interlacing being therefore indispensable, we find again a single representative opening for the table:

1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13
	2 4 6	2 5 7	2 8 10	2 9 12	2 11 13

This is unchanged by the group of order 48: $G = \{ (4\,5)\,(6\,7), (4\,6)\,(5\,7), (4\,7)\,(5\,6); (8\,11\,12)\,(9\,10\,13), (8\,9)\,(10\,12)\,(11\,13); (1\,2)\,(4\,5\,7\,6)\,(8\,9\,12\,13\,11\,10) \}$.

Two cases now arise: 4 7 3 and 4 7 8. The former is invariant under (4 7) (5 6). Its continuations are

4 8 11	4 9 13	4 10 12	7 8 11	7 9 13	7 10 12
4 8 12	4 9 11	4 10 13	7 8 12	7 9 11	7 10 13
4 8 13	4 9 10	4 11 12	7 8 13	7 9 10	7 11 12
4 8 13	4 9 11	4 10 12	7 8 13	7 9 11	7 10 12

where again any row on the right goes with any row on the left.

The first three rows are equivalent under the substitution (8 11 12) (9 10 13), and the last row on the left is incompatible with any row on the right. There remain then only two possibilities

4 8 11	4 9 13	4 10 12	7 8 12	7 9 11	7 10 13
			7 8 13	7 9 10	7 11 12

and these are equivalent (8 9) (10 12) (11 13). Further triads must now be formed from 8 with 3, 5, 6, 13 and from 13 with 3, 5, 6, 8, an obvious conflict. The case 4 7 3 is therefore to be rejected.

3. The case 4 7 8 is invariant under (1 2) (5 6) (9 10) (11 12). There are three subcases 4 3 9, 4 3 11, and 4 3 13, and these lead to

4 3 9	4 10 13	4 11 12	7 3 11	7 9 13	7 10 12	*a*
			7 3 13	7 9 11	7 10 12	*b*
4 3 11	4 9 13	4 10 12	7 3 9	7 10 13	7 11 12	*c*
			7 3 12	7 9 11	7 10 13	*d*
			7 3 13	7 9 10	7 11 12	*e*
4 3 13	4 9 10	4 11 12	7 3 11	7 9 13	7 10 12	*f*
			7 3 12	7 9 11	7 10 13	*g*
4 3 13	4 9 11	4 10 12	7 3 9	7 10 13	7 11 12	*h*
			7 3 10	7 9 13	7 11 12	*i*

Here *a* and *c*, *b* and *h*, *e* and *f* are equivalent under (4 7) (5 6); *f* and *g*, *h* and *i* are equivalent under (1 2) (5 6) (9 10) (11 12). Further, *a* and *b* are equivalent under (1 4) (3 8) (2 7) (10 13 12 11). And finally *d* requires triads to be formed from 9 with 3, 5, 6, 10 and 10 with 3, 5, 6, 9.

There survive only two distinct cases:

I.	4 3 9	4 10 13	4 11 12	7 3 11	7 9 13	7 10 12,
II.	4 3 11	4 9 13	4 10 12	7 3 13	7 9 10	7 11 12.

Case II is continued by 12 3 8, 12 5 6, and either of the following:

11 5 8	11 6 9	9 3 5	10 3 6	10 5 13	13 6 8,
11 6 8	11 5 9	9 3 6	10 3 5	10 6 13	13 5 8.

These last are equivalent under (6 12) (1 2 11 3 13 4 7 9 8 10).

The complete triad system for Case II is then:

1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13
2 1 3	2 4 6	2 5 7	2 8 10	2 9 12	2 11 13
3 1 2	3 4 11	3 5 9	3 6 10	3 7 13	3 8 12
4 1 5	4 2 6	4 3 11	4 7 8	4 9 13	4 10 12
5 1 4	5 2 7	5 3 9	5 6 12	5 8 11	5 10 13
6 1 7	6 2 4	6 3 10	6 5 12	6 8 13	6 9 11
7 1 6	7 2 5	7 3 13	7 4 8	7 9 10	7 11 12
8 1 9	8 2 10	8 3 12	8 4 7	8 5 11	8 6 13
9 1 8	9 2 12	9 3 5	9 4 13	9 6 11	9 7 10
10 1 11	10 2 8	10 3 6	10 4 12	10 5 13	10 7 9
11 1 10	11 2 13	11 3 4	11 5 8	11 6 9	11 7 12
12 1 13	12 2 9	12 3 8	12 4 10	12 5 6	12 7 11
13 1 12	13 2 11	13 3 7	13 4 9	13 5 10	13 6 8

Here the interlacing is as follows:

1 with 2, 9, 10, 12;	2 with 1, 5, 8, 9;	3 with 7, 9, 10;
4 with 7, 11, 12, 13;	5 with 2, 6, 7, 12;	6 with 5, 10, 13;
7 with 3, 4, 5, 11;	8 with 2, 11, 13;	9 with 1, 2, 3, 13;
10 with 1, 3, 6, 11;	11 with 4, 7, 8, 10;	12 with 1, 4, 5;
13 with 4, 6, 8, 9.		

The group which leaves the triad system unchanged is therefore intransitive and does not connect 1, 2, 4, 5, 7, 9, 10, 11, 13 and 3, 6, 8, 12. It contains the substitutions (1 2 9) (3 12 8) (4 11 7) (5 3 10) and (1 11) (2 4) (5 13) (7 9) (8 12), but not any substitution (3 6 8) $\cdots$ nor any that leaves 3, 8, 12 each fixed. It follows that the group is of order 6 and is that generated by the two substitutions above. The systems of transitivity are 1, 2, 4, 7, 9, 11; 3, 8, 12; 5, 10, 13; 6.

4. Case I leads to

9 5 10	9 6 11	10 3 6	11 5 8	6 5 12	6 8 13	5 3 13	8 3 12	*a*
				6 5 13	6 8 12	5 3 12	8 3 13	*b*
9 5 11	9 6 10	10 3 5	11 6 8	6 5 12	6 3 13	5 8 13	8 3 12	*c*
				6 5 13	6 3 12	5 8 12	8 3 13	*d*

Of these *b*, *c*, *d* are equivalent to Case II under (1 5 2 6) (4 7) (3 12 9 11 10 13) (8), (1 9 7 8 10 6) (2 3 5 12 13 4) (11), (1 3 6 7 13 2 10 12) (8 4 9 11) (5), respectively. There remains only *a*, which leads to the triad system

1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13
2 1 3	2 4 6	2 5 7	2 8 10	2 9 12	2 11 13
3 2 1	3 4 9	3 5 13	3 6 10	3 7 11	3 8 12
4 1 5	4 2 6	4 3 9	4 7 8	4 10 13	4 11 12
5 1 4	5 2 7	5 3 13	5 6 12	5 8 11	5 9 10
6 1 7	6 2 4	6 3 10	6 5 12	6 8 13	6 9 11
7 1 6	7 2 5	7 3 11	7 4 8	7 9 13	7 10 12
8 1 9	8 2 10	8 3 12	8 4 7	8 5 11	8 6 13
9 1 8	9 2 12	9 3 4	9 5 10	9 6 11	9 7 13
10 1 11	10 2 8	10 3 6	10 4 13	10 5 9	10 7 12
11 1 10	11 2 13	11 3 7	11 4 12	11 5 8	11 6 9
12 1 13	12 2 9	12 3 8	12 4 11	12 5 6	12 7 10
13 1 12	13 2 11	13 3 5	13 4 10	13 6 8	13 7 9

The interlacing is as follows:

1 with 2, 3, 4, 5, 12, 13;
2 with 1, 3, 8, 10, 11, 13;
3 with 1, 2, 4, 6, 9, 10;
4 with 1, 3, 5, 7, 8, 9;
5 with 1, 4, 6, 8, 11, 12;
6 with 3, 5, 9, 10, 11, 12;
7 with 4, 8, 9, 10, 12, 13;
8 with 2, 4, 5, 7, 10, 11;
9 with 3, 4, 6, 7, 11, 13;
10 with 2, 3, 6, 7, 8, 12;
11 with 2, 5, 6, 8, 9, 13;
12 with 1, 5, 6, 7, 10, 13;
13 with 1, 2, 7, 9, 11, 12.

The system is invariant under the substitutions

(1 2 8 4 3 10 7 9 6 12 13 11 5) and (2 4 12) (3 5 13) (8 7 10) (6 11 9).

It is readily found that no substitution with two fixed letters leaves the system invariant. The order of its group is therefore a divisor of 13.12, and it is metacyclic. In fact its order is 39.

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TRIPLE-SYSTEMS AS TRANSFORMATIONS, AND THEIR PATHS AMONG TRIADS*

BY

H. S. WHITE

A triple-system has been of interest as offering a puzzling problem in construction. When various methods of construction have given systems apparently different, means of comparison have to be devised. The group of substitutions that transform a triple-system into itself has been adopted as a convenient abstract mark or character for discrimination; since certainly two systems whose groups are abstractly different cannot belong to similar triple-systems. I propose here a more direct means of comparison of two triple-systems, a method which incidentally facilitates the discovery of the group. This method consists in regarding the triple-system itself as a symbol of operation, and deducing certain covariants of that operation. These covariants can be represented in some respects graphically. I here apply such graphic representation to the well-known triple-systems on thirteen elements; thirteen being the lowest number of elements for which there exist two dissimilar or incongruent triple-systems.

As in some other transformations, so here also the inverse problem of invariance presents difficulties, the question, namely, what relations between the invariants of two operations are sufficient to prove the equivalence of the operations themselves. In the case here presented, where the possible operations (triple-systems on thirteen elements) are only two in number, the answer is immediate and complete; but I do not attempt any generalization.

Reciprocity or duality in a finite set mediated by a triple-system.

Recall the definition of a triple-system. It consists of triads of the chosen elements, so numerous and so related that while every pair of elements (or *dyad*) occurs in some triad, no pair occurs in two triads. Thus if there are 13 elements, every element occurs with 6 pairs of others, and there are in all 26 triads.

It is precisely this property that fits the triple-system to be a transformer of

* Presented to the Society, September 11, 1912.

dyads into single elements. Since every dyad occurs once and no more, this duality will be unique for dyads; but its inverse would transform a single element into 6 dyads. *Let us consider solely the transformation of a dyad into an element, and the resulting transformation of triads into triads.*—Every combination of three elements comprises three dyads; and when each of these becomes an element, the triad becomes a new triad. Thus if the triple system contain the three triads 124, 135, 236, then it will transform the triad 123 into 456.

From 13 elements, 286 triads can be formed. Any proposed triple-system contains 26 of these, leaving 260 that we may call extraneous triads. Apply the triple-system to transform them all; the 26 in the system are all self dual, i. e., they are transformed into themselves, but the 260 extraneous triads go partly into extraneous, partly perhaps into triads of the system. All that are found to be produced by the transformation we may call *derivative*, all that are missing after the transformation we shall call primitive. By trial on the two types of triple-systems, REISS'S and NETTO'S, we find that the former leaves out 104 primitive triads, the latter 130.

This distinction of primitive and derivative is obviously not affected if the triple-system and all extraneous triads be themselves subjected to any operation of substitution upon the 13 elements. Hence the distinction is invariant under the smaller group of substitutions which transform the triple-system into itself. We have then, under this group, three invariant sets of triads: those in the system, the extraneous-derivative triads, and the extraneous-primitives. Other subdivisions will appear soon.

Iterated duality. Trains of triads.

Under the dual transformation of a given triple-system Δ, let a triad t_1 be converted into another triad t_2. Repeat the operation; let t_2 become t_3. Continue the iteration indefinitely. Since only 286 triads exist, one of two things must happen: either a triad of the system is reached, say t_k; or else a triad that has appeared already is repeated, say $t_{m+n} \equiv t_m$. In the former case the triad t_k repeats forever, while in the latter case the train beginning from t_m constitutes an ever recurring cycle. Both cases may be included in one if we will speak of the triads of the system as one-term cycles. We can formulate the statement that *every triad that is primitive with respect to a given triple-system initiates a train terminating in a periodic cycle.* Presumably any one cycle may have several initial primitive triads, though it is conceivable that it might be made up exclusively of derivatives. If there are in a train primitives and derivatives that do not recur in the terminal cycle, they may be classified as forming *appendices*; then every complete train will be made up of one recurrent cycle and all its appendices. In the two examples to be pre-

sented there are appendices both short and long, some consisting of a single primitive triad, some of a primitive and seven derivatives.

The totality of the periodic cycles constitutes a system of triads, invariant under the automorphic group of the triple-system and also invariant under the dual transformation mediated by the triple-system itself.

The totality of complete trains (cycles with their appendices) constitutes a system of triads invariant under those substitutions on the 13 *elements that transform the triple-system into itself, and covariant with the triple-system under the symmetric group on the* 13 *elements.*

The Reiss triple-system on 13 *elements.*

The first to write down the triples of a system on 13 elements was REISS, whose system may be arranged for convenience in the following rectangular array. Each element heads one column, and below are placed the 6 pairs of elements that occur with it in triads of the system.

1	2	3	4	5	6	7	8	9	*a*	*b*	*c*	*d*
23	13	12	56	46	45	26	15	16	59	47	14	17
58	67	49	28	18	19	35	24	34	78	29	79	25
69	48	57	39	37	27	1*d*	7*a*	5*a*	36	38	68	89
4*c*	5*d*	6*a*	1*c*	2*d*	3*a*	4*b*	9*d*	2*b*	1*b*	1*a*	2*a*	4*a*
7*d*	9*b*	8*b*	7*b*	9*a*	8*c*	8*a*	3*b*	7*c*	2*c*	5*c*	5*b*	6*b*
ab	*ac*	*cd*	*ad*	*bc*	*bd*	9*c*	6*c*	8*d*	4*d*	6*d*	3*d*	3*c*

A triad extraneous to this system is for example 246. In column 2 we note the pair *48* and the pair 67, in column 4 the pair 65, and have thus found the transformed triad *578*. This in turn transforms into 31*a*, etc. The process is simple and not too tedious, and we find eight classes of trains, as follows:

I. Six classes terminating in triads of the system:

(1) Three with no appendix, 169, 79*c*, and 89*d*. |—|

(2) Four with two single appendices. Example

$$\begin{matrix}235\\4ab\end{matrix}\ 17d. \quad >|—|$$

(3) Six with a double appendix. Example,

$$37a \quad 568 \quad 14c\ . \quad — — |—$$

(4) Nine with double bifid appendix. Example,

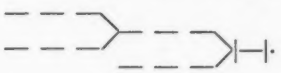
$$\begin{matrix}46b\\89a\end{matrix}\quad 57d \quad 123\ . \quad >— |—$$

(5) One with single trifid appendix, viz.,

$$\begin{matrix}145\\27a\\3bd\end{matrix}\quad 68c \qquad \geqslant |—$$

(6) Three with two appendices, one quadruple and one octuple bifid. Example:

$$\begin{array}{cccc} 16d & 79b & 24c & 18a \\ 18d & 795 & 3ac & 26d \end{array} \Big\rangle \begin{array}{cccc} 57b & 34c & 19d & 678 \\ 689 & 1cd & 347 & 59b \end{array} \Big\rangle 2ac$$

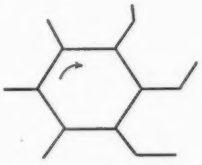


II. Two classes terminating in cycles of period 6.

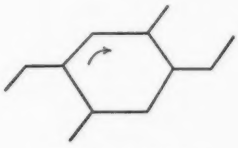
(7) Six whose appendices in order contain 1, 1, 1, 2, 2, 2 triads. Example: one whose initial primitives are $7ad$, 468, 259, $35c$, $3ab$, 134.

(8) Six whose appendices in order contain 1, 2, 0, 1, 2, 0 triads. Example: one whose initial primitives are 457, 149, $12a$, 279.

Graphs showing the order of succession of these latter two classes are the most noteworthy new result brought out by this method. For a rough sketch we may omit the elements of the triads, and denote each triad by a segment of a line, make a primitive end where a derivative begins, arrange the cycle as a regular hexagon, and insert an arrow to point the direction of derivation. Appendices fall outside, and their extreme segments are the primitives of the trains.



Class 7, 6 trains.



Class 8, 6 trains.

Netto's triple-system on 13 *elements.*

The second kind of triple-system on 13 elements is NETTO'S, given here in the same tabulation as REISS'S.

1	2	3	4	5	6	7	8	9	a	b	c	d
$2b$	$3c$	$4d$	15	26	37	48	59	$6a$	$7b$	$8c$	$9d$	$1a$
ad	$1b$	$2c$	$3d$	14	25	36	47	58	69	$7a$	$8b$	$9c$
45	56	67	78	89	$9a$	ab	bc	cd	$1d$	12	23	34
79	$8a$	$9b$	ac	bd	$1c$	$2d$	13	24	35	46	57	68
38	49	$5a$	$6b$	$7c$	$8d$	19	$2a$	$3b$	$4c$	$5d$	16	27
$6c$	$7d$	18	29	$3a$	$4b$	$5c$	$6d$	17	28	39	$4a$	$5b$

When triads are transformed through this table, the resulting trains fall into 3 classes of 13 each. Two take the place of the first six of the Reiss case; and the entirely extraneous trains are a single class, all similar, instead of two sixes.

Triads of the system terminate 26.

(1) Thirteen have a double appendix. Example: 49*b*, 236, 57*c*. — — |—|.

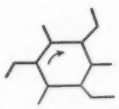
(2) Thirteen have three simple appendices.
Example:

18*b*
49*d* 23*c*. ≻|—|
567

External triads form also a third class.

Class (3). Thirteen terminate in cycles of period 6, with appendices in order containing 1, 2, 1, 2, 1, 2, triads. Example: one whose primitives are

7*cd*, 17*b*, 346, 68*c*, 59*a*, 2*ac*.



Class 3, 13 trains.

These graphs render the dissimilarity of the two systems obvious.

The trains exhibit the groups.

Substitutions on the 13 elements will transform every train into one precisely similar. Therefore any operation of the group that leaves a triple-system invariant can only change any particular train into itself, or into another of the same class. To find that group, then, we may seek first substitutions that transform into itself one of the largest trains, then those that transform it into each one in succession of its own class. Substitutions thus found must be tested on all trains of the class simultaneously, then upon the other classes, and only those retained which conserve every class. Or when found they may be tested directly upon the triple-system itself.

For the Netto Δ_{13} this is quite simple. Looking at the graphs of the third class, we surmise that each has cyclic substitutions into itself of period 3, and that each may be transformed into any one of the other twelve—i. e., that these transformations, with the identity, constitute a sub-group of order 13. Barring a third conceivable style of change, this would give us a total group of order 39.

The other possibility is namely this, that some substitution on the elements might convert every triad contained in a train into itself; i. e., that the identity on triads might result from some substitution not the identity on the elements. The larger the train, the easier the inspection that discards this hypothesis;

for any letter that is found in two triads having the other letters different must go over into itself if both triads are to go over into themselves. That conceivable third style of substitutions is found to be impossible in either set of trains.

We shall give special detailed attention to the group of the Reiss Δ_{13}. For this purpose it will be shortest to use the six trains of Class 7. None of them has any substitution into itself; and we shall find that the requirement of changing the first into any other particular train determines the substitution completely. In writing these trains, let a bar at the left of a triad denote a primitive, and let a dash between two indicate that the one at the right is derived from that at the left through the Reiss Δ_{13}. The six trains of Class 7 are the following, with appendices both above and below the line of the repeating cycle of six.

$$
\begin{array}{lccccccccccc}
|23a & - & 16c & & |37d & & |56c & & |49b & & & \\
\cdots & 156 & - & 489 & - & 23d & - & 15c & - & 48b & - 237 - & 156 \cdots \\
|15a & - & 89b & & & & & & |48a & - & 27d & \\
\end{array}
$$

$$
\begin{array}{lccccccccccc}
|245 & - & 68d & & |127 & & |368 & & |9ab & & & \\
\cdots & 38d & - & 9bc & - & 257 & - & 36d & - & abc & - 125 - & 38d \cdots \\
|34d & - & 9ac & & & & & & |4bc & - & 157 & \\
\end{array}
$$

$$
\begin{array}{lccccccccccc}
|35b & - & 78c & & |15d & & |28c & & |49a & & & \\
\cdots & 27c & - & 69a & - & 135 & - & 278 & - & 46a & - 35d - & 27c \cdots \\
|27b & - & 469 & & & & & & |6ab & - & 13d & \\
\end{array}
$$

$$
\begin{array}{lccccccccccc}
|3ab & - & 168 & & |7ad & & |468 & & |259 & & & \\
\cdots & 146 & - & 59c & - & 7ab & - & 148 & - & 25c & - abd - & 146 \cdots \\
|134 & - & 29c & & & & & & |35c & - & 7bd & \\
\end{array}
$$

$$
\begin{array}{lccccccccccc}
|24a & - & 8cd & & |147 & & |8bc & & |359 & & & \\
\cdots & 8bd & - & 369 & - & 14a & - & bcd & - & 356 & - 47a - & 8bd \cdots \\
|2bd & - & 569 & & & & & & |236 & - & 17a & \\
\end{array}
$$

$$
\begin{array}{lccccccccccc}
|45b & - & 67c & & |1bd & & |6ac & & |239 & & & \\
\cdots & 7ac & - & 289 & - & 4bd & - & 67a & - & 238 & - 14b - & 7ac \cdots \\
|57a & - & 389 & & & & & & |258 & - & 14d & \\
\end{array}
$$

To transform the first of these trains into the second, for example, evidently we must transform each triad into one in the same vertical column, and indeed into the one lying in the corresponding horizontal line. In the second column from the left we see that 1, occurring three times, must go over into d; similarly in the sixth column 4 must go over into b. Again in column two, lines one and two, 6 goes into 8, and so c, 5 into 6, 3 respectively; and in line three, a into 4. Exactly in the same way from column six we find that b, 9, 8 go over into a, 9, c

respectively. This leaves for determination 2, 3, 7, d. Of these, 2, 7 in column four are seen to go over into 5, 1 respectively, while the other two are found conveniently in column seven, where we have 3, d converted into 2, 7. The complete substitution is thus determined, as was to be proven. It is, condensed in usual form,

$$S_{12} \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & a & b & c & d \\ d & 5 & 2 & b & 3 & 8 & 1 & c & 9 & 4 & a & 6 & 7 \end{pmatrix}$$

$$\equiv (9)(1d7)(253)(4ba)(68c).$$

Indeed this has for a corollary the non-transformability of the second train into itself. Looking further, we verify that S_{12} changes the third into the first, the second into the third, and permutes cyclically also the fourth, fifth, and sixth trains of this class. But when we once have the substitution in hand, it is shorter to prove it directly upon the tabular form of the Δ_{13} at once; and there we find the entire columns permuted with the same substitution as the elements.

It is necessary further only to find a substitution, $S_{14} \equiv (2b)(3a)(45)(7d)(8c)$, that converts the first train into the fourth of Class 7, and that with S_{12} generates the group of order 6. This being well known as the group of this Δ_{13}, the method receives confirmation, and need not be carried out in further detail.

Conclusion.

Attention having been given hitherto chiefly to the construction of triple-systems, the time has come for new methods of comparison and description, as contributing to facility in classification.

By the analogue of projections and linear transformations, applied under the concept of finite geometries, we are led to regard the triple-system as an operator. Like the quantic in symbolic algebra, it has an indefinitely extensive range of objects on which to act. In this sketch, we have applied it to triads, and found that the triple-system Δ_{13} operating on all triads of its elements arranges them in trains, most of which are characteristic of the system; and that especially the trains composed exclusively of triads extraneous to the system are an aid in the discovery of the group to which the system belongs. In the light of existing knowledge of the two species of Δ_{13}'s we can formulate theorems whose independent proof is not yet found. One example is sufficient:

THEOREM: *If any triple-system on* 13 *letters contains a triad not derivable, through the duality mediated by the system, from any extraneous triad, then the triple-system is of the Reiss species and possesses three such triads.*

One may examine sets of 13 triads that are invariant under the group of the Netto triple-system. Of these there are but two, the primitives and first

derivatives in the 13 trains of the first class. These two together constitute a second Netto triple-system, equivalent of course to the first This, with the recollection that the group arranges all triads in conjugate sets, makes evident the theorem:

Triple-systems of the Netto type on 13 *elements occur in pairs, the two of a pair having the same group but having no triad in common. The triads in such a pair can combine by alternating thirteens to form in one way a second pair of triple-systems related to each other in the same way as the first pair.* Of interest is the fact that if the two systems of one pair be employed to transform each other, the other pair appears; and a second transformation with the new systems restores the old in inverted order.

PROOF OF POINCARÉ'S GEOMETRIC THEOREM

BY

GEORGE D. BIRKHOFF *

In a paper recently published in the Rendiconti del Circolo Matematico di Palermo (vol. 33, 1912, pp. 375–407) and entitled *Sur un théorème de Géométrie*, POINCARÉ enunciated a theorem of great importance, in particular for the restricted problem of three bodies; but, having only succeeded in treating a variety of special cases after long-continued efforts, he gave out the theorem for the consideration of other mathematicians.

For some years I have been considering questions of a character similar to that presented by the theorem and it now turns out that methods which I have been using are here applicable. In the present paper I give the brief proof which I have obtained, but do not take up other results to which I have been led.†

1. Statement of the Theorem. Poincaré's theorem may be stated in a simple form as follows: Let us suppose that a continuous one-to-one transformation T takes the ring R, formed by concentric circles C_a and C_b of radii a and b respectively $(a > b > 0)$, into itself in such a way as to advance the points of C_a in a positive sense, and the points of C_b in the negative sense, and at the same time to preserve areas. *Then there are at least two invariant points.*

In the proof of this theorem we shall use modified polar coördinates $y = r^2$, $x = \theta$ where r is the distance of the point (x, y) from the center of the circles, and θ is the angle which a line from the center to (x, y) makes with a fixed line through the center. The transformation T may be written then

$$x' = \varphi(x, y), \qquad y' = \psi(x, y).$$

The function $\psi(x, y)$ is a continuous function of (x, y), uniquely determined at each point of R, and so is periodic in x of period 2π. The function $\varphi(x, y)$ admits of an infinite number of determinations which differ from each

* Presented to the Society, October 26, 1912.

† Some of my results are contained in a paper entitled *Quelques théorèmes sur les mouvements des systèmes dynamiques*, which is shortly to appear in the Bulletin de la Société Mathématique de France.

other by integral multiples of 2π, and these determinations can be grouped so as to form continuous branches. Since $(x + 2\pi, y)$ and (x, y) represent the same point of R, the algebraic difference between the values of one of these determinations taken for $(x + 2\pi, y)$ and (x, y) reduces to an integral multiple of 2π; and this difference must be one and the same multiple of 2π for all x and y because the difference is a continuous function. But if the point (x, y) makes a positive circuit of the circle C_a, the same is true of its image (x', y'); hence along this path $\varphi(x, y)$ increases by 2π when x increases by 2π. Thus the difference reduces identically to 2π; in other words, the function $\varphi(x, y)$ increases by 2π when x is increased by 2π.

In consequence of these properties of $\varphi(x, y)$ and $\psi(x, y)$, it is clear that $x' - x$ and $y' - y$ are both single-valued and continuous in R.

The precise meaning of the theorem is that if any determination of $\varphi(x, y)$ is made for which

$$x' > x \text{ along } C_a \quad \text{and} \quad x' < x \text{ along } C_b,$$

(the conditions on T make possible such a choice) then we shall have at least two points (x, y) of R for which

$$x' = x, \qquad y' = y.$$

2. On the Method of Proof. As Poincaré remarks (loc. cit., p. 377), the existence of one invariant point implies immediately the existence of a second invariant point. *Hence if the theorem is false we may assume that there is no point invariant for T.* In this case we shall have

$$(x' - x)^2 + (y' - y)^2 > d^2 > 0 \tag{1}$$

for all points of R since $x' - x$ and $y' - y$ are single-valued, continuous, and not simultaneously zero, over the ring R. We shall establish the theorem by proving that the hypothesis (1) admits a *reductio ad absurdum.*

3. The Auxiliary Transformation. If $0 < \epsilon < b^2$, the one-to-one continuous transformation T_ϵ given by

$$x' = x, \qquad y' = y - \epsilon$$

takes the circles C_a and C_b into the concentric circles $C_a' : y = a^2 - \epsilon$ and $C_b' : y = b^2 - \epsilon$ respectively, so that C_a' is within C_a at a distance $a - \sqrt{a^2 - \epsilon}$ from C_a, and C_b' is similarly within C_b at a distance $b - \sqrt{b^2 - \epsilon}$ from C_b. This transformation effects a shrinking of the plane toward the origin which leaves every point on its radius vector and which preserves areas, since dx and dy are unaltered by the transformation and the integral of areas is

$$\int\int r\, dr\, d\theta = \tfrac{1}{2} \int\int dx\, dy.$$

Now, as long as $\epsilon < d$, the auxiliary transformation TT_ϵ, formed by following T by T_ϵ, also has no invariant point. For *if we take x and y as the rectangular coördinates of a point in the strip S:*

$$-\infty < x < +\infty, \qquad b^2 \leqq y \leqq a^2,$$

corresponding to the ring R, we see that the point (x, y) is displaced at least d units in S by T and then is further displaced a distance ϵ by T_ϵ in the direction of the negative y-axis, so that if $\epsilon < d$, the point cannot come back to its initial position. We write this compound transformation TT_ϵ in the form

$$\bar{x}' = \varphi(x, y), \qquad \bar{y}' = \psi(x, y) - \epsilon.$$

Let us choose the positive quantity ϵ once for all and so small that

$$\epsilon < b^2, \qquad \epsilon < d, \qquad \epsilon < a^2 - b^2. \tag{2}$$

Consider now the multiple-valued function

$$\omega(x, y) = \arctan \frac{y' - y}{x' - x}, \tag{3}$$

or, more accurately, those branches of this function which give the angle that the vector drawn from the point (x, y) to the point (x', y') in the strip S makes with the positive direction of the x-axis (the other branches corresponding to the negative of this vector). In virtue of (1), this function is continuous at every point of S and accordingly falls into branches single-valued and continuous throughout S.

Moreover any such branch reduces to a fixed even and odd multiple of π along C_a and C_b respectively, since along C_a we have $x' > x$, $y' = y$ and along C_b, $x' < x$, $y' = y$.

The functions $y' - y$ and $x' - x$ have been seen to be periodic in x of period 2π, and so any such branch of $\omega(x, y)$ differs at $(x + 2\pi, y)$ and (x, y) by a multiple of 2π which may reduce to zero. Since the branches are continuous, this multiple is one and the same throughout S. But along C_a and C_b these branches have a constant value, as we have noted. Hence the multiple will in fact reduce to zero. Thus these branches of $\omega(x, y)$ are periodic in x of period 2π, i. e., are single-valued in R.

Likewise it is clear that the multiple-valued function

$$\bar{\omega}(x, y) = \arctan \frac{\bar{y}' - y}{\bar{x}' - x}, \tag{4}$$

which gives the angle that the vector drawn from (x, y) to $(\bar{x}', \bar{y}')$ makes with the positive direction of the x-axis, falls into single-valued and continous branches. Moreover the functions $\bar{y}' - y = y' - y - \epsilon$ and $\bar{x}' - x = x' - x$ are

periodic in x of period 2π; also each of the branches of $\bar{\omega}(x, y)$ is periodic in x along C_a and along C_b. We conclude thatthe branches of $\bar{\omega}(x, y)$ as well as of $\omega(x, y)$ are periodic in x of period 2π, i. e., are single-valued in R.

The branches of $\omega(x, y)$ and $\bar{\omega}(x, y)$ may be associated in pairs so that the maximum numerical difference of any pair for every (x, y) is regulated in accordance with the formula

$$|\omega(x, y) - \bar{\omega}(x, y)| < \frac{\pi}{2}. \tag{5}$$

This is obvious geometrically, for the point (x', y') is at least d units distant from the point (x, y) in S, and $(\bar{x}', \bar{y}')$ is distant $\epsilon < d$ units from (x', y') so that the angle subtended at (x, y) by these last-named points can never become equal to $\frac{1}{2}\pi$. Thus if we associate the branches at a single point of S in accordance with (5), this inequality must continue to be true as the coordinates of the point vary continuously in S.

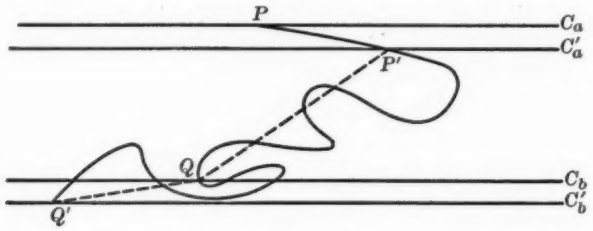
4. Construction of a Curve Invariant for the Auxiliary Transformation. The transformation TT_ϵ of R takes C_a to the circle C_a' of radius $\sqrt{a^2 - \epsilon}$, *between* C_a and C_b, since we chose $\epsilon < a^2 - b^2$. The repetition of TT_ϵ takes C_a' into C_a'', a simple closed curve. Now TT_ϵ is a one-to-one and continuous transformation of the ring R into the ring R', limited by C_a', C_b' and obtained by letting R shrink under the transformation T_ϵ. Since C_a' lies within R and encloses C_b, its image C_a'' must lie within R' (and accordingly within C_a') and enclose C_b'. Furthermore the transformation TT_ϵ takes the ring formed by $C_a C_a'$ in R into a second ring $C_a' C_a''$ in R', which abuts on the inner boundary C_a' of the first ring. If C_a'' lies wholly in R it will enclose C_b as well as C_b', and the image C_a''' of C_a'' by TT_ϵ will be a simple curve within C_a'' and enclosing C_b'. Thus $C_a'' C_a'''$ will form the boundary of a third ring abutting on the inner boundary of the second ring $C_a' C_a''$. This process may be continued to form simple curves C_a', C_a'', $\cdots$ lying one within the other, and corresponding rings $C_a C_a'$, $C_a' C_a''$, $\cdots$, so long as the curves continue to lie wholly in R and not in part or wholly within C_b.

The area of the ring $C_a C_a'$ is $\pi\epsilon$ and will, *since the transformations T and T_ϵ each preserve areas*, be the same as that of its image rings $C_a' C_a''$, $C_a'' C_a'''$, $\cdots$. This series of rings can only terminate when a curve $C_a^{(n)}$ is reached ($n \geqq 2$) part, at least, of which lies within the circle C_b. But for every l, the area included between C_a and $C_a^{(l)}$ is $\pi l\epsilon$, which if l is sufficiently large exceeds the area of R. It follows that such a curve $C_a^{(n)}$ exists. Hence we can find a particular point P of C_a whose nth image lies inside of the circle C_b.

Now let us turn to the strip S and to one of the representations of P on the upper side C_a of the strip. Here C_a' is represented by a straight line at distance ϵ below C_a, and C_a'', C_a''', $\cdots$ are each represented by a simple open curve

congruent by sections $2k\pi \leqq x \leqq 2(k+1)\pi$ and extending indefinitely to right and left. The rings $C_a C'_a$, $C'_a C''_a$, $\cdots$ clearly are represented by the successive strata between successive curves of this set in S. Let P' be the first image of P and join PP' by a straight line which will lie entirely within the strip $C_a C'_a$. Let the arc $P'P''$ be the image of PP' under TT_ϵ, and let $P''P'''$ be the image of $P'P''$ under the same transformation, and so on. In this way we construct successively PP', $P'P''$, $\cdots$, $P^{(n-1)}P^{(n)}$ lying respectively in the strata $C_a C'_a$, $C'_a C''_a$, $\cdots$, $C_a^{(n-1)} C_a^{(n)}$ in S, the end point $P^{(n)}$ of the last of these arcs falling below the lower side C_b of S.

Let Q be the first intersection of this succession of arcs with the lower side



C_b of the strip (see figure). It is obvious that the curve PQ formed by this succession of arcs is a simple curve, for PP', $P'P''$, $\cdots$ are successive simple arcs which lie in the successive strata $C_a C'_a$, $C'_a C''_a$, $\cdots$ in S. Furthermore PQ lies wholly between C_a and C_b.

The image $P'Q'$ of PQ under the transformation TT_ϵ is made up of $P'Q$ and an extended simple arc QQ', the image of the arc $Q^{-1}Q$ where Q^{-1} is the point that goes into Q by TT_ϵ. The arc QQ' being the image of points of R by TT_ϵ lies wholly below the straight line C'_a; the end-point Q' of this arc lies of course C'_b. Furthermore QQ' has no point but Q in common with PQ. For if such a point exists it must lie on $P'Q$, and by performing the transformation inverse to TT_ϵ we see that $Q^{-1}Q$ has a point other than Q^{-1} in common with PQ^{-1}, which is not possible since PQ has no multiple points. Thus PQ' forms a simple curve.

The transformation TT_ϵ takes the arc PQ of PQ' into the arc $P'Q'$ of PQ', advancing each point of PQ along PQ'. In this sense the curve PQ is *invariant* under the transformation.

The properties of T and T_ϵ ensure that P' has an x greater than that of P, and that Q' has an x less than that of Q, as indicated in the figure.

5. The Rotation of the Auxiliary Point-Image Vector on the Invariant Curve. If a point B moves along PQ' from P to Q, it is clear that its image $\bar{B}'$ by TT_ϵ moves from P' to Q' along the same curve, never coinciding with B. We shall now establish the fact (intuitively almost self-evident) that the

corresponding rotation of the vector $B\bar{B}'$ thus obtained is $-\pi$ plus the sum of the two acute angles which the straight lines PP' and QQ' make with the x-axis.

At the outset it is obvious that the rotation can differ from this value only by a multiple of 2π. That the rotation has precisely the value stated depends entirely on simple considerations of *analysis situs*. The fact on which this conclusion rests is that a continuous deformation of the curve $PP'QQ'$ through a series of simple curves containing P, P', Q, Q' brings the curve to the broken line position $PP'QQ'$ (see figure).

Let t be any monotonic parameter for the curve PQ taking on the increasing values t_0, t_0', t_1, t_1', at P, P', Q, Q' respectively. Let $\tau(t)$ be the value of this parameter for $\bar{B}'$ where t is its value for the corresponding point B. Clearly $\tau(t)$ is a continuous increasing function of t $(t_0 \leqq t \leqq t_1)$ which has the property $\tau(t) > t$. Consider now any varied simple curve through the same four points P, P', Q, Q' on which a monotonic parameter t is so chosen that as before t_0, t_0', t_1, t_1' correspond to P, P', Q, Q' respectively.* If the distance between any point of the varied curve and the point of PQ with the same parameter value is uniformly small, the corresponding vector $B\bar{B}'$ along the varied curve will undergo precisely the same total rotation as along PQ, since the initial and final positions are the same in either case and the angular differences of the vectors in all intermediate positions are uniformly small also.

As a consequence of this reasoning it follows that we may deform the curve $PP'QQ'$ continuously through any series of simple curves containing the same four points P, P', Q, Q' provided that for the varied curves the parameter is properly chosen, and the total rotation of $B\bar{B}'$ will not thereby be altered.

The curve PQ lies wholly in S, so that the straight line QQ', being outside of S, does not intersect the curve PQ. It is therefore apparent that the arc QQ' may be continuously deformed to this straight line position QQ' without taking the curve QQ' outside of the strip formed by C_a' and C_b'; for the continuum formed by this strip is simply-connected after a cut in it by the curve $P'Q$ is made, and hence any simple curve joining the points Q and Q' of this continuum, and lying in it, can be continuously deformed into any other such simple curve through a series of simple arcs not having any points but Q and Q' in common with the boundary of the cut strip.

Next, the arc $P'Q$ may be continuously deformed, through a series of simple arcs joining P' to Q in the strip $C_a'C_b$ in which $P'Q$ wholly lies, into the straight line PQ, inasmuch as the straight lines QQ' and PP' have no points within this strip.

*I include in the term "varied simple curve" any simple curves with the same geometric locus but with a different parameter.

Hence the rotation of $B\bar{B}'$ during its first series of positions is precisely the same as it is along the broken line formed by the three straight segments PP', $P'Q$, QQ'. The rotation along the broken line from the initial to the final position is clearly $-\pi$ plus the sum of the acute angles which the vectors PP' and QQ' make with the x-axis. Thus our statement is proved.

In the first series of positions, $\bar{B}'$ is obtained from B by the transformation TT_ϵ. If the coördinates of B are (x, y), those of $\bar{B}'$ are accordingly $(\bar{x}', \bar{y}')$, and *the rotation of $B\bar{B}'$ is measured by the change in the function $\bar{\omega}(x, y)$ as (x, y) moves from P to Q along the invariant curve PQ.*

Let us fix upon that continuous branch $\bar{\omega}_1(x, y)$ of this function which is measured at the point P by the negative of the acute angle which the vector PP' makes with the x-axis. At the point Q this determination will have the value $-\pi$ plus the acute angle which the vector QQ' makes with the x-axis.

6. The Rotation of the Point-Image Vector for T. Now fix upon that continuous branch $\omega_1(x, y)$ of the function $\omega(x, y)$ which, along the upper side C_a of S, takes on the value zero.

These two functions $\bar{\omega}_1(x, y)$ and $\omega_1(x, y)$ differ by less than $\frac{1}{2}\pi$ at the point P and hence are branches of $\bar{\omega}(x, y)$ and $\omega(x, y)$ associated by the inequality (5), which holds throughout S. Moreover the terminal value of the function $\bar{\omega}_1(x, y)$ has been shown to differ from $-\pi$ by less than $\frac{1}{2}\pi$ at Q, so that $\omega_1(x, y)$ differs from $-\pi$ by less than π at Q. Any branch of the function $\omega(x, y)$ however has been seen to be precisely equal to a fixed odd multiple of π along C_b. Hence this function $\omega_1(x, y)$ must have the value $-\pi$ at Q and also at all points of C_b.

The variation of $\omega_1(x, y)$ is therefore $-\pi$ when the point (x, y) moves in any manner whatever from a point of C_a to a point of C_b. In other words *if we let the point B move in any manner from C_a to C_b in S, and let B' denote the image of B by the transformation T, the total rotation in the vector BB' will be precisely $-\pi$.*

7. Completion of the Proof. Consider now the transformation T^{-1} inverse to T, which is in every respect similar to T except that points on C_a and C_b, and hence on the two sides of the strip S, are moved in the reverse direction. By symmetry, the vector $B'B$ which joins B' to its image B under T^{-1} must now rotate through an angle $+\pi$ as B and B' move from C_a to C_b.

Here the vector $B'B$ is of sense opposite to that of BB'. But the actual rotation of the vectors BB' and $B'B$ is of course one and the same quantity, so that a contradiction has been reached. Hence the theorem is completely proved.

8. Generalizations of the Theorem. In phrasing his geometric theorem Poincaré makes the hypothesis that we have *some* integral invariant $\iint P(x, y)\,dx\,dy$ $(P(x, y) > 0)$, not necessarily the invariant of areas. However, we can,

by a suitable change of coördinates from (x, y) to (ξ, η), change the integral invariant to the simpler area invariant.

Let the lines $\eta =$ const. be the circles concentric with C_a and C_b, and let the number $\eta(y)$ be so chosen for each circle that as a point moves from C_b to C_a the double integral taken over the ring between C_b and the concentric circle through the moving point is equal to η:

$$\eta(y) = \int_{b^2}^{y} \left\{ \int_0^{2\pi} P(x, y)\, dx \right\} dy.$$

This function $\eta(y)$ is clearly eontinuous and increasing and has a positive continuous derivative, namely

$$\frac{d\eta(y)}{dy} = \int_0^{2\pi} P(x, y)\, dx.$$

Speaking somewhat inexactly, the η-curves are all so placed as to measure off equal increments of $\iint P(x, y)\, dx\, dy$ for equal increments of η. Now (speaking in the same inexact sense) choose the ξ-curves so as to measure off equal increments of the same double integral between these successive η-curves and the line $x = 0$.

The possibility of making such a choice of curves $\xi =$ const. may be seen as follows: Let $x = f(y)$ be any curve lying in R, joining C_a to C_b, and such that the part of the area between the initial line $x = 0$, this curve, the circle C_b and the circle $y =$ const. is constantly proportional to $\eta(y)$:

$$\int_{b^2}^{y} \left\{ \int_0^{f(y)} P(x, y)\, dx \right\} dy = \frac{k}{2\pi} \eta(y).$$

In view of the above mentioned properties of $\eta(y)$, this equation is equivalent to

$$\int_0^{f(y)} P(x, y)\, dx = \frac{k}{2\pi} \frac{d\eta(y)}{dy}. \tag{6}$$

For a given k, the quantity $f(y)$ is clearly a single-valued continuous function of y since $P(x, y)$ and $d\eta(y)/dy$ are positive and continuous. Moreover an increase in k continuously increases $f(y)$. Thus; we get a set of non-intersecting curves $x = f(y)$ which, like the circles $\eta =$ const., fill up the ring R. It is necessary to notice that $x = 0$ corresponds to $k = 0$, and that $x = 2\pi$ is the curve for $k = 2\pi$ by definition of $\eta(y)$. We take k as determined by the above method to be the coördinate ξ of any point on the curve $x = f(y)$.

Thus we have coördinates (ξ, η) for which the integral invariant is the invariant $\iint d\xi d\eta$. Taking (ξ, η) as the modified polar coördinates of a point in a new plane, we find that, expressed in these coördinates, T has all the

properties specified for T in the theorem. Thus we infer that there are at least two invariant points as before.

A further generalization is that the curves C_a and C_b may be allowed to be any simply closed curves, one within the other, bounding a ring R; we may again state a similar theorem, for we can make a preliminary transformation, for instance a conformal one, to take these curves into concentric circles, when of course the integral invariant merely changes form. Here it may be necessary to consider with care the nature of the integral invariant near the boundaries after the transformation of coördinates.

Finally we may permit the function $P(x, y)$ to vanish at some or all points of the curves C_a and C_b. Under certain restrictions it is certain that in this limiting case there will be invariant points also.

9. Poincaré's Method. It is interesting to notice that Poincaré uses only a single property consequent on the existence of an invariant integral, namely that no continuum on R can be transformed into part of itself by the transformation T (loc. cit., p. 377). It seems improbable that this condition is equivalent to the condition that there exists an invariant integral. The existence of such an invariant integral is a fact which enters more intimately into the proof which I have given above. I do not know whether the modification of Poincaré's theorem which results when the condition that an integral invariant exists is replaced by this weaker condition is true.

HARVARD UNIVERSITY,
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ON THE EXISTENCE OF LOCI WITH GIVEN SINGULARITIES*

BY

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Introduction.

1. The numerical relations existing between ordinary or so-called *Plückerian* singularities of a plane curve were determined as early as 1834 by PLÜCKER, but the inverse question has been left almost untouched. It may be stated thus: *To show the existence of a curve having assigned Plückerian characters*; and is equivalent to the determination of the maximum of cusps κ_M that a curve of order m and genus p may have. VERONESE † has solved the question for rational curves. As an example of errors that have been committed in this direction, we cite SALMON–FIEDLER'S ‡ statement that a seventhic may have 13 cusps, in which case however it is found that it would have (-2) bitangents!

In this paper a condition derived from the theory of invariants is given for the characteristics of a curve, a discussion of the *a priori* possible curves follows, and the existence of the curves within a certain range is then shown.

Finally a few words will be said as to the extension of the theory to surfaces and higher varieties.

Absolute invariants of a curve.

2. Let $e_1, e_2, \cdots, e_R$ be the coefficients of a quantic Q in x_1, x_2, x_3, and $e_1', e_2', \cdots, e_R'$ be the coefficients of Q', a quantic in (x_1', x_2', x_3') obtained from

* Presented to the Society, September 12, 1911.

† *Uber die Methode des Projicirens und Schneidens*, Mathematische Annalen, vol. 19 (1882), p. 209.

‡ *Höhere Ebene Kurven*, p. 83.

Q by the transformation:

$$x_i = \sum_{k=1}^{3} \lambda_{ik} x'_k \qquad (i=1, 2, 3).$$

The equations of transformation are:

$$e'_j = f_j(e, \lambda) \qquad (j=1, 2, \cdots, R)$$

linear in the (e)'s and rational in the λ's, with no other denominator than some power of the determinant Δ of the transformation which we will suppose not to vanish. Following ANDOYER * let us consider the array:

$$\left\| \begin{matrix} \frac{\partial f_1}{\partial \lambda_{11}}, & \frac{\partial f_1}{\partial \lambda_{12}}, & \frac{\partial f_1}{\partial \lambda_{13}}, & \frac{\partial f_1}{\partial \lambda_{21}}, & \cdots, & \frac{\partial f_1}{\partial \lambda_{33}} \\ \frac{\partial f_2}{\partial \lambda_{11}}, & \frac{\partial f_2}{\partial \lambda_{12}}, & \cdots & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial f_R}{\partial \lambda_{11}}, & \cdots, & \cdots & & & \end{matrix} \right\|$$

and suppose that at least one of the determinants of order r ($r \leqq 9$) that can be derived from it does not vanish, while all those of order $r+1$ do. Then the system can be reduced to an equivalent one formed of r equations containing the (λ)'s, and $(R-r)$ freed from them.† The latter will have to be satisfied by the coefficients of any quantic derived from Q by a linear transformation of non-vanishing determinant. The left members of these $(R-r)$ equations will be *absolute invariants* of Q. If (e') and (e'') are any two quantics, the necessary and sufficient conditions for the passage from one to the other by a linear transformation of non-vanishing Δ, is that

$$F_j(e') = F_j(e'') \qquad (j = 1, 2, \cdots R-r),$$

the (F)'s being $(R-r)$ algebraically independent absolute invariants. In general

$$R = \frac{(m+1)(m+2)}{2}$$

and $r = 9$. If $r < 9$, then the (λ)'s are functions of $(9-r)$ arbitrary parameters, from which follows at once that Q is invariant under a continuous projective group, hence the curve represented by $Q = 0$ is of genus 0 or 1 (Theorem of Schwarz). Hence: *A curve of genus greater than one depends upon* $\frac{(m+1)(m+2)}{2} - 9$ *independent absolute invariants.* Unless otherwise stated *we shall suppose in what follows that* $p \geq 2$.

* *Leçons sur la théorie des formes*, p. 14.

† With possibly additional equations of restrictive character.

Conditions for a double point or a cusp.

3. It has been shown by V. SNYDER,* that for a given order m, curves exist for each genus up to the maximum $\frac{(m-1)(m-2)}{2}$. From this follows that the system of relations (E) satisfied by the coefficients of the general ternary quantic of order m, when its representative curve has δ double points, cannot be reduced to the system of relations (E') corresponding to a curve with $(\delta - 1)$ double points. Since (E') is part of (E), it follows that the latter contains at least one more equation than (E'), that is: *the addition of a double point increases the number of relations satisfied by the coefficients by at least one unit.* If we now consider the most general unicursal curve of order m, it can be represented by the system:

$$x_i = \sum_{k=0}^{m} a_k^i t^{m-k} \qquad (i = 1, 2, 3),$$

which contains $3m + 2$ non-homogeneous arbitraries. But the transformations

$$t = \frac{\alpha\theta + \beta}{\gamma\theta + 1}$$

are the only ones that leave the curve unchanged, and since this group contains 3 parameters, it follows that there are $\infty^{3m+2-3} = \infty^{3m-1}$ rational curves of order m. Hence if the general ternary quantic of order m represents one of these curves, its coefficients have to satisfy

$$\frac{m(m+3)}{2} - (3m-1) = \frac{(m-1)(m-2)}{2}$$

relations, which number is precisely the number of double points then imposed upon the curve. Hence, *each double point reduces the number of arbitraries by just one unit.*

In what precedes we suppose of course that the only conditions imposed refer to the number of double points, and that, for instance, their position is left arbitrary. This arbitrariness however is not always unrestricted. Thus it is not enough to state that a sextic of deficiency 1 has 9 double points, but it must also be said then that these are not on a cubic; for if not so, the latter counted twice would give the sextic required. Such an additional condition however is not expressed by a relation but by an inequality, and furthermore involves the coördinates of the double points in question.

4. Let us now consider the number of relations indicating the presence of κ cusps. Since the system (E_1) indicating that there are δ double points

* *Construction of curves of given deficiency.* Bulletin of the American Mathematical Society, vol. 15 (1908), p. 3.

and κ cusps of assigned position, has only κ more relations than the system (E) corresponding to $(\delta + \kappa)$ double points, certainly when κ double points are changed into cusps no more than κ relations between the coefficients alone are thereby introduced. Let this number be $\kappa - \lambda$, λ will depend upon m, p, κ. Whenever we can show that there are curves of order m_0, genus p_0, with either κ_0 or $(\kappa_0 + 1)$ cusps, we shall have

$$\lambda(m_0, p_0, \kappa_0) = \lambda(m_0, p_0, \kappa_0 + 1),$$

since the systems of relations for κ_0 and $\kappa_0 + 1$, between the coefficients of the curve, contain a different number of relations. Also $\lambda(m_0, p_0, \kappa_0) = 0$. It follows that if we show that for given m, p, the curve can have successively $1, 2, \cdots, \kappa_1$ cusps, then $\lambda(m, p, \kappa) = 0$ for $\kappa \leqq \kappa_1$. In a discussion to follow we will admit as a postulate that $\lambda = 0$, and we will say then that the *postulate of singularities is satisfied.* As we just saw, it is equivalent to the admission that when a curve with characters (m, p) can have κ_1 cusps it can also have any number of cusps smaller than κ_1. Of course this is almost evident, but the possibility of a proof is great, and the postulate in question can be accepted temporarily, pending further increase of knowledge in this direction. As we shall see later, this proposition will be verified within quite a wide range. Meanwhile let us show that *it is true when* $\kappa \leqq m - 2$ or that a curve *may acquire successively as many as* $(m - 2)$ *cusps.* For this purpose consider the equation:

$$x_3 \prod_{i=1}^{n-1} (x_1 + \alpha_i x_2) + \varphi_m(x_1, x_2) = 0,$$

φ_m being a binary $m$thic. If we make

$$\alpha_1 = \alpha_2 = \cdots = \alpha_{k-1} = \alpha_k \qquad (k \leq m - 1),$$

this equation will represent a special rational curve of order m having at $(0, 0, 1)$ a point of multiplicity $m - 1$, at which k of the $m - 1$ tangents coincide, which is equivalent to $\dfrac{(m-1)(m-2)}{2} - (k - 1)$ double points and $(k - 1)$ cusps. Here clearly every new cusp adds a new relation.

With regard to curves of the type

$$\sum_{h=k}^{m} x_3^{m-h} \varphi_h(x_1, x_2) = 0$$

it is interesting to remark that when the multiplicity of $(0, 0, 1)$ is raised from k to $k + 1$, we have to satisfy

$$\frac{(k+1)(k+2) - k(k+1)}{2} = k + 1$$

relations, while only k double points have been added. Hence *if we increase the multiplicity of a point by h without changing the genus, we decrease the number of arbitraries by h.*

The maximum number of cusps.

5. In what follows we shall assume the truth of the postulate of singularities as stated in § 4. According to this postulate an additional cusp imposes two conditions upon the coefficients of the curve. Each of these conditions will be expressed by the vanishing of an invariant, since both are satisfied by any linear transformed of the curve considered. More properly speaking and with reference to § 2, the absolute invariants of the curve will have to satisfy two new relations. Referring then to the same section we infer that *a curve of order m cannot have more than* $\left[\frac{m(m+3)}{4} - 4\right]$ *cusps*, with the reservation that its genus shall then be greater than one, and this, as will be seen, is actually the case. If $m \equiv 0$ or $1 \pmod 4$ this number is equal to $\frac{m(m+3)}{4} - 4$, and the curves with the maximum number of cusps can have no other double point. If $m \equiv 2$ or $3 \pmod 4$ this number is equal to $\frac{m(m+3)-2}{4} - 4$ and besides the cusps the corresponding curve can have one more double point.

6. The *a priori* possible curves will evidently have positive Plückerian characters, and we are thus led *to discuss Plücker's equations in order to find the maximum of cusps alone such that i and τ are positive or zero*, i and τ being the number of inflexions and bitangents. We shall use only the following three formulæ:

$$n = m(m-1) - 2\delta - 3\kappa, \qquad (n = \text{class})$$

$$i = 3m(m-2) - 6\delta - 8\kappa,$$

$$2(\tau - \delta) = (n-m)(n+m-9).$$

If we suppose $\delta = 0$ they become:

$$n = m(m-1) - 3\kappa,$$

$$i = 3m(m-2) - 8\kappa,$$

$$2\tau = (n-m)(n+m-9);$$

from which we deduce:

$$n - m = m(m-2) - 3\kappa,$$

$$n + m - 9 = m^2 - 9 - 3\kappa,$$

hence:

$$2\tau = [m(m-2) - 3\kappa][m^2 - 9 - 3\kappa].$$

The first factor on the right is inferior to the second if $m(m-2) \lesseqgtr m^2-9$, or if $m > 4$; and certainly if we suppose, as we shall do, $m \geq 7$. Hence $\tau \geq 0$ if

$$\kappa \lesseqgtr \left[\frac{m(m-2)}{3}\right].$$

It is seen at once that when this condition is fulfilled both i and n are positive. Hence *a curve has certainly less than* $\frac{m(m-2)}{3}$ *cusps.* We have to compare this last limit to the one found previously. For this purpose we consider

$$\frac{m(m-2)}{3} - \left(\frac{m(m+3)}{4} - 4\right) = \frac{(m-3)(m-14)+6}{12}.$$

This difference is negative if $m < 14$. Hence for $m \geq 14$ we must use the limit found in the preceding paragraph, while for $m \lesseqgtr 13$ we must use the limit given above.

We have supposed $m \geq 7$, since all the questions considered in this paper have been solved long ago for $m < 7$. We may remark however that the quintic with 5 cusps * and the sextic with nine cusps find their place in this discussion. From the above we must not be surprised however that neither the tricuspidal quartic nor the cubic with one cusp satisfy the condition of this section.

Discussion of the general problem.

7. We shall now discuss the problem stated in the introduction, assuming of course the truth of the postulate of singularities. If we introduce p in them, the formulæ of Plücker used in § 6 can be written as follows:

$$\begin{aligned}
n &= m(m-1) - 2(\delta+\kappa) - \kappa \\
&= m(m-1) - (m-1)(m-2) + 2p - \kappa \\
&= 2(m+p-1) - \kappa, \\
i &= 3(m-2)m - 6(\delta+\kappa) - 2\kappa = 3m(m-2) - 3(m-1)(m-2) \\
&\qquad + 6p - 2\kappa \\
&= 3(m+2p-2) - 2\kappa, \\
2\tau &= 2\delta + (n-m)(n+m-9) \\
&= (m-1)(m-2) - 2p - 2\kappa + (m+2p-2-\kappa)(3m+2p-11-\kappa) \\
&= \kappa^2 - [4(m+p)-11]\kappa + 4[(m+p)^2 - 5m - 7p + 6] \\
&= \varphi(\kappa;\ m,\ p).
\end{aligned}$$

* This is DEL PEZZO's quintic. See DEL PEZZO: *Sulla quintica con cinque punti cuspidale*, Napoli Rendiconti, series 2, vol. 3 (1889), p. 46.

Since we must have only positive or zero characteristics we obtain at once the following conditions:

(1) $$\kappa \leqq 2(m+p-1) = \alpha,$$

(2) $$\kappa \leqq \tfrac{3}{2}(m+2p-2) = \beta,$$

(3) $$0 \leqq \varphi(K;\ m,\ p).$$

The postulate of singularities gives another inequality when $p \geq 2$:

$$\delta + 2\kappa \leqq \frac{m(m+3)}{2} - 8,$$

or

$$\frac{(m-1)(m-2)}{2} - p + \kappa \leqq \frac{m(m+3)}{2} - 8,$$

and finally:

(4) $$\kappa \leqq 3m + p - 9 = \gamma \qquad (p \geq 2).$$

If the postulate is not assumed we have to replace (4) by

(4′) $$\kappa \leqq 3m + p - g + \lambda(m,\ p) = \gamma',$$

but in this case of course the discussion is difficult.

Let κ_1 and κ_2 be the roots of $\varphi(\kappa;\ m,\ p) = 0$ considered as a quadratic in κ. We must compare then the numbers $\alpha, \beta, \gamma, \kappa_1$ and κ_2 and in this will consist the discussion. The roots of $\varphi(\kappa;\ m,\ p)$ are real if

$$[4(m+p) - 11]^2 - 16[(m+p)^2 - 5m - 7p + 6]$$
$$= -8m + 24p + 25 \geq 0,$$

or if

$$m \leqq 3(p+1).$$

Also

$$\kappa_1 \kappa_2 = 4[(m+p)^2 - 5m - 7p + 6]$$
$$= (m+p)(m+p-5) - 2(p-3) > 0$$

if $m > 2$, since then

$$m + p - 5 > p - 3,$$
$$m + p > 2.$$

Finally:

$$\kappa_1 + \kappa_2 = 4(m+p) - 11 > 0 \quad \text{if} \quad m \geq 3.$$

Hence *if* $m \geq 3$, *the roots of* $\varphi(\kappa;\ m,\ p)$ *are either positive or imaginary.* As to α, β, γ we have:

$$\alpha - \beta = \tfrac{1}{2}m - (p-1) \geq 0 \quad \text{if} \quad m \geq 2(p-1),$$
$$\alpha - \gamma = -m + p + 7 \geq 0 \quad \text{if} \quad m \leqq p + 7,$$
$$\beta - \gamma = \tfrac{3}{2}m + 2p + 6 \geq 0 \quad \text{if} \quad m \leqq \tfrac{4}{3}p + 4.$$

If $2 \leqq p \leqq 9$ we have $0 < 2p - 2 \leqq \frac{4}{3}p + 4 \leqq p + 7$.
If $p = 9$ the 3 limits coincide.
If $p > 9$ we have

$$0 < p + 7 \leqq \tfrac{4}{3}p + 4 \leqq 2p - 2,$$

and $3p + 3$ is of course always above all these limits.

8. *Suppose first* $p \geq 9$. We have to consider several cases.

Case 1. $$6 < m \leqq p + 7.$$

Then $\gamma < \alpha < \beta$ and it is γ that we must compare to κ_1 and κ_2. We have

$$\varphi(\gamma; m, p) = (3m + p - 9)[3m + p - 9 - 4m - 4p + 11] + 4[(m + p)^2 - 5m - 7p + 6]$$
$$= (m - p)^2 - 5m + p + 6.$$

Let $m - p = 9$, then $-p < 9 < 7$ in the case considered. Also

$$\varphi(\gamma; m, p) = q^2 - 5q - 4p + 6 = F(q).$$

If q_1 and q_2 are the roots of $F(q)$ $(q_1 \leqq q_2)$, we must locate them with respect to $-p$ and 7. Since

$$q_1 q_2 = -4p + 6 < 0 \qquad q_1 < 0 < q_2,$$

also

$$F(-p) = p^2 + p + 6 > 0$$

always, and

$$F(7) = 20 - 4p < 0$$

in the case considered. Hence

$$-p < q_1 < 0 < 7 < q_2,$$

and q_1 is the only root in the interval considered. Accordingly if

$$-p < q < q_1, \qquad \varphi(\gamma) > 0, \qquad \gamma \leqq \kappa_1;$$

but if $$q_1 < q < 7, \qquad \varphi(\gamma) < 0, \qquad \kappa_1 \leqq \gamma.$$

As

$$q_1 + p = \frac{5 - \sqrt{16p + 1}}{2} \quad \text{and} \quad +p = \frac{2p + 5 - \sqrt{16p + 1}}{2}$$

we may conclude as follows:

If $m < \dfrac{2p + 5 - \sqrt{16p + 1}}{2}$ we must have $\kappa \leqq \gamma$.

If $m > \dfrac{2p + 5 - \sqrt{16p + 1}}{2}$, $\kappa \leqq \kappa_1$.

Case 2. $$p + 7 \leqq m \leqq 2p - 2.$$

Then of the numbers $\alpha, \beta, \gamma, \alpha$ is the smallest and is therefore to be compared to κ_1 and κ_2. We have

$$\varphi(\alpha; m, p) = 2(m+p-1)[2(m+p-1) - 4m - 4p + 11] + 4[(m+p)^2 - 5m - 7p + 6]$$
$$= -2[21m + 25p - 6] < 0.$$

In this interval therefore we have again $\kappa_1 < \alpha$ and the condition $\kappa \equiv \kappa_1$.

Case 3. $2p - 2 \equiv m$.

Then β is smaller than either α or γ, and is to be compared to κ_1 and κ_2. We have:

$$4\varphi(\beta; m, p) = 3(m+2p-2)[3(m+2p-2) - 2(4(m+p) - 11)] + 16[(m+p)^2 - 5m - 7p + 6] = (m-2p-1)^2 - 8p + 1$$
$$= (m - 2p - 1 - \sqrt{8p+1})(m - 2p - 1 + \sqrt{8p+1}) < 0$$

in the interval considered, if $m < 2p + 1 + \sqrt{8p+1}$, since here $2p - 2 > 2p + 1 - \sqrt{8p+1}$ for $p > 9$. Finally:

$$\beta - \tfrac{1}{2}(\kappa_1 + \kappa_2) = -\tfrac{1}{2}m - p + \tfrac{9}{2} < 0 \quad \text{for} \quad p > 9.$$

Therefore if $2p - 2 \equiv m \equiv 2p + 1 + \sqrt{8p+1}$, then $\kappa_1 \equiv \beta$ and therefore we must have $\kappa \equiv \kappa_1$; while if $2p + 1 + \sqrt{8p+1} < m$, $\beta < \kappa_1$ and the condition is $\kappa \equiv \beta$. This completes the discussion when $p \geq 9$.

9. Suppose now $2 \equiv p \equiv 9$.

Case 1. $m \equiv \tfrac{4}{3}p + 4$.

Then γ is smaller than α or β, and is to be compared to κ_1. As in the preceding paragraph we find:

$$\varphi(\gamma) = q^2 - 5q - 4p + 6 = E(q);$$

with $-p < q < \tfrac{1}{3}p + 4$. Now

$$F(\tfrac{1}{3}p + 4) = (\tfrac{1}{3}p + 4)(\tfrac{1}{3}p + 4 - 5) - 4p + 6 = \tfrac{1}{9}(p^2 - 27p + 36).$$

The parenthesis has for roots 25.6 and 1.4, hence here

$$F(\tfrac{1}{3}p + 4) < 0,$$

and accordingly

$$-p < q_1 < 0 < \tfrac{1}{3}p + 4 < q_2.$$

Hence if

$$m < \frac{2p + 5 - \sqrt{16p+1}}{2} < 6, \qquad \kappa \equiv \gamma;$$

and if

$$\frac{2p+5-\sqrt{16p+1}}{2} < m < \tfrac{4}{3}p+4, \qquad \kappa \geqq \kappa_1.$$

Case 2. $\qquad \tfrac{4}{3}p+4 < m.$

Then β is smaller than either α or γ.
We find as in section 8 that

$$\varphi(\beta;\ m,\ p) = (m-2p-1+\sqrt{8p+1})\,(m-2p-1-\sqrt{8p+1}),$$

and it can be verified easily that

$$2p+1-\sqrt{8p+1} < \tfrac{4}{3}p+4 < 2p+1+\sqrt{8p+1}$$

when $p \geqq 9$. If then $\frac{4}{3}p+4 \geqq m \geqq 2p+1+\sqrt{8p+1}$, $\varphi(\beta) < 0$ and $\kappa \geqq \kappa$, is the condition; while if $2p+1+\sqrt{8p+1} \geqq m$, then $\varphi(\beta) > 0$ and we must have $\kappa \geqq \beta < \kappa_1$.

We may remark that if $m > 6$ and $p \geqq 1$, $m > 2p+1+\sqrt{8p+1}$, and the condition $\kappa \geqq \beta$ is also necessary. We shall see that it is also sufficient for the existence of the corresponding curves.

10. The preceding discussion can be tabulated as follows:

$$\begin{array}{r|l} m & 7 \cdots 2p+1+\sqrt{8p+1} \cdots +\infty \\ p \geqq 9 & \kappa \geqq \kappa_1, \qquad \kappa \geqq \frac{3}{2}(m+2p-2) \end{array},$$

$$\begin{array}{r|l} m & 7 \cdots \dfrac{2p+5-\sqrt{16p+1}}{2} \cdots 2p+1+\sqrt{8p+1} \cdots +\infty \\ p > 9 & \kappa \geqq 3m+p-9, \qquad \kappa \geqq \kappa_1, \qquad \kappa \geqq \frac{3}{2}(m+2p-2) \end{array}.$$

The value of κ_1 is:

$$\kappa_1 = \tfrac{1}{2}\left[4(m+p) - 11 - \sqrt{-8m+24p+25}\right].$$

If any of the values given in the table exceeds $\dfrac{(m-1)(m-2)}{2} - p$ it simply means that all double points can *a priori* become cusps for the system (m, p) considered. It may be remarked that the only case in which the postulate of singularities affects the discussion is when

$$7 \geqq m \geqq \frac{2p+5-\sqrt{16p+1}}{2} \qquad \text{and} \qquad p > 9,$$

for if it is not admitted to be true, we have

$$\gamma = 3m+p-9+\lambda(m, p),$$

and it is only in this case that of the numbers α, β, γ, the latter is the smallest.

11. The following problem is also interesting: *how many more double points or cusps can be added to a curve of order m above δ double points and κ cusps.* The preceding discussion is of no avail inasmuch as p is indefinite in this case. We shall see that this problem is suggested by the theory of surfaces.

We shall not carry the discussion into any details and shall merely state that for a concrete value of m, the best procedure is to plot curves giving κ in terms of p in

$$2(m+p-1)-\kappa=0,$$

$$3(m+2p-2)-2\kappa=0,$$

$$\varphi(\kappa;\, m,\, p)=0, \qquad 3m+p-9-\kappa=0$$

and then to find the region of the plane for which the left members are positive.

The same discussion can of course be applied to other singularities than cusps. Here however we are forcibly reminded of the fact that our discussion defines only *upper limits* which may or may not be reached. For example let us propose to determine the largest number of quadruple points that a seventhic can have. To have one quadruple point there must be 8 relations between its independent absolute invariants. The number of the latter is $7.10/2-8=27$, from which would follow that a seventhic may have as many as three quadruple points, but of course even if it has as many as two the curve will break up into a sextic and a straight line.

Existence of the curves.

12. While we have been unable to give positive existence proofs covering the whole range of the tables of § 10, we shall see that the range actually covered is quite extensive.

The following preliminary lemma is quite important. *If for given (m, p) all curves for which $\kappa \leqq \kappa_0$ exist, the same holds for the curves having the characters m, p', κ' such that $p' \geq p, \kappa' \leqq \kappa_0$.* For we may take the general equation of curves of the mth order, and make it represent a curve with characters (m, p, κ). As coefficients and the coördinates of these singular points will have to satisfy a certain system E composed of a certain number of equations and inequalities. If $p' \geq p$, $\kappa' \leqq \kappa$, it is clear that the system E' corresponding to (m, p', κ') is *contained in the system* E, by which we express the fact that the equations and inequalities of E' are respectively among the equations and inequalities of E. Any solution of E will be a solution of E', but not the only one, for then E could be reduced to a simpler system, in which case some of the double points or cusps of the curve corresponding to (m, p, κ) would be the consequence of others, but this is against the supposition that for the values (m, p), curves exist for all values of κ inferior to κ_0, and also

against the fact that double points are independent (§ 3). Our lemma is thus proved.

13. We shall now proceed to show that *a curve with characters* (m, p, κ), *for which the table in* § 10 *gives the condition* $\kappa \leqq \frac{3}{2}(m + 2p - 2)$, *exists if* κ *is inferior to this limit.* Referring to the tables this means that if

$$m \geq 2p + 1 + \sqrt{8p + 1}$$

and the curve has positive characters, its existence can be proved. For then

$$m > 2(p - 1) + 1 + \sqrt{8(p - 1) + 1},$$

and therefore the table will give for the curves having the characters $(m, p - 1, \kappa)$ the condition:

$$\kappa \leqq \tfrac{3}{2}[m + 2(p - 1) - 2].$$

Suppose now that the proposition is proved for the curve corresponding to $(m, p - 1, \kappa)$, then by our lemma the curve corresponding to (m, p, κ) can have as many as $[\frac{3}{2}[(m + 2(p - 1) - 2)]]$ cusps. Suppose that

$$\kappa = [\tfrac{3}{2}m] + 3(p - 2) + h; \quad \text{and} \quad 1 \leqq h \leqq 3,$$

then will

$$h = 2(m + p - 1) - \left[\frac{3}{2}m\right] + 3(p - 2) + h = \frac{m + e}{2} - (p + h - 4),$$

$$i = 3(m + 2p - 2) - 2\left[\frac{3}{2}m\right] - 6(p - 2) - 2h = e + 2(3 - h),$$

with $e = 0$ or 1 according as m is odd or even. Now (n, p, i) are the characters of the reiprocal of the curve considered. It is clear that $i \leqq 5$, hence either $n \leqq 6$, — and then since the reciprocal has positive characters its existence is certain; or $n > 6$, and then our lemma applies (since a curve having characters $(n, 0, \kappa)$ can have as many as $n - 2$ cusps (§ 4), and $n - 2 \geq 5$ here), it follows that a curve of order n and genus p can certainly have as many as 5 cusps when $n > 6$. In both cases the reciprocal exists certainly, and the same holds for the curve with characters (m, p, κ); this proves the proposition considered, as soon as it is proved for $p = 0$. This however has been done by Veronese (loc. cit., § 1), and we shall also give a proof later.

Let us now find the limiting values of p, κ for which the theorem holds. We must suppose m given and solve the equation

$$m \geq 2p + 1 + \sqrt{8p + 1}$$

for p. It includes:

$$m > 2p + 1 \quad \text{or} \quad p < \frac{m - 1}{2},$$

which remark after a simple calculation shows that

$$p \leqq \frac{m+3-\sqrt{3\,(2m+3)}}{2};$$

then

$$\beta = \tfrac{3}{2}\,m + 3p - 3 = 3\,(m+\tfrac{1}{2}) - \tfrac{3}{2}\,\sqrt{3\,(2m+3)},$$

and this is the highest value of κ, the existence of which is certain. Remarking that the proposition can be applied when

$$n \geq 2p+1+\sqrt{8p+1},$$

we obtain a new extension of the limits found. This will occur if

$$2\,(m+p-1) - \kappa \geq 2p+1+\sqrt{8p+1},$$

or if $0 < \kappa \leqq 2m-3-\sqrt{8p+1}$, that is, *for given* (m, p) *a curve can have as many as* $1, 2, \cdots, (2m-3-\sqrt{8p+1})$ *cusps.*

14. There is a class of special cuspidal curves for which an equation can be given readily. Let $X=0$, $Y=0$ represent two straight lines not passing through any of the vertices of the triangle of reference. Then

$$xyz\,(aX+bX)^{m-3} + X^m = 0$$

is the equation of a curve having a contact of order $(m-1)$ at the intersection of $X=0$ with the sides of the triangle of reference and also a point of multiplicity $(m-3)$ with as many coincident tangents at the intersection of $X=0$ with $Y=0$. This multiple point is equivalent to $(m-4)$ cusps and a certain number of double points. If we apply now the transformation:

$$x'x = y'y = z'z,$$

we obtain

$$\frac{1}{x'y'z'}\left(\frac{a'}{x'}+\frac{b'}{y'}+\frac{c'}{z'}\right)^{m-3},+\left(\frac{d'}{x'}+\frac{e'}{y'}+\frac{f'}{z'}\right)^m = 0.$$

This is a curve of order $m'=2m$, having four points of multiplicity m, m, m, $m-3$ respectively at each of which the tangents all coincide. The equivalent number of cusps is $3\,(m-1)+m-4 = 4m-7 = 2m'-7$. To obtain curves of odd order consider:

$$xyz\,(aX+bY)^{m-3} + X^{m-1}\,(X+cx) = 0.$$

The same tranformation will give, if $X+cx \equiv dy+ez$, the following equation:

$$\left(\frac{a'}{x'}+\frac{b'}{y'}+\frac{c'}{z'}\right)^{m-3} + x'\left(-\frac{c}{x'}+\frac{d}{y'}+\frac{e}{z'}\right)^{m-1}(dz'+ey') = 0,$$

which represents a curve of order $m'=2m-1$, with four points of multi-

plicity $m-1$, $m-1$, m, $m-3$, respectively, at each of which all tangents are coincident. The equivalent number of cusps is: $2(m-2)+m-1+m-4=4m-9=2(2^m-1)-7=2m'-7$ again. In both cases

$$p=\frac{(m-1)(m-2)}{2}-\frac{(m-3)(m-4)}{2}=2m-5=2\left[\frac{m'+1}{2}\right]-5.$$

If instead of $(aX+bY)^{m-3}$ we had taken $(aX+bY)^{m-3-h}\varphi_h(x, y, z)$, we should have obtained the following two equations:

$$\frac{1}{x'y'z'}\left(\frac{a'}{x'}+\frac{b'}{y'}+\frac{c'}{z'}\right)^{m-3-h}\varphi_h\left(\frac{1}{x'}, \frac{1}{y'}, \frac{1}{z'}\right)+\left(\frac{d'}{x'}+\frac{e'}{y'}+\frac{f'}{z'}\right)^{\frac{m'}{2}}=0 \qquad (m' \text{ even}),$$

$$\frac{1}{x'y'z'}\left(\frac{a'}{x'}+\frac{b'}{y'}+\frac{1'}{z'}\right)^{\frac{m-5-h}{2}}\varphi_h\left(\frac{1}{x'}, \frac{1}{y'}, \frac{1}{z'}\right)+\left(\frac{d'}{x'}+\frac{e'}{y'}+\frac{f'}{z'}\right)^{\frac{m'-1}{2}}\left(\frac{e'}{y'}+\frac{f'}{z'}\right)=0 \qquad (m' \text{ odd}),$$

representing curves having $(2m'-h-7)$ cusps, with

$$h \leqq \left[\frac{m'+1}{2}\right]-3.$$

We merely indicate here that by making the curve $\varphi_h(x, y, z)=0$ have a certain number of multiple points on $dx+cy+fz=0$ the above curves can be made to have a genus varying within a certain range. It would not be uninteresting to find whether p can be made to have the same value when h changes; it is no doubt possible within certain limits.

15. We have already stated that curves of order inferior to 6 are well known to exist when their characteristics are positive. The same may be stated for the case $m=6$, but as no one to the best of our knowledge has considered them under this light, and as it is important for what follows, we shall briefly consider sextics, and show *the existence of any sextic with positive characters.*

(1) *Rational sextics.* The proof of their existence follows the same lines as the one which will be given subsequently. A rational sextic has at most $3(6-2)=6$ cusps. A sexticuspidal sextic has for reciprocal a quartic, a quinticuspidal sextic has for reciprocal a quintic, so that their existence is certain. Next, the equation

$$z(x-\alpha y)^h(x-\beta_1 y)\cdots(x-\beta_{5-h}y)+\varphi_6(x, y)=0$$

represents a rational sextic having a singularity equivalent to $5-h$ cusps and $5+h$ double points.

(2) *Elliptic sextics.* In this case all double points may become cusps. The sextics with 9, 8, 7 cusps are inverse of a cubic, quartic, quintic, respectively. If we consider next the triangle ABC formed by three inflexional tangents to

a cubic, and if we apply to the curve the quadratic transformation of § 14 we obtain a sextic having three triple points with coincident tangents. Each of these triple points is equivalent to 2 cusps, hence we have obtained a particular case of the curve $(6, 1, 6)$. If BC were merely tangent to the cubic or absolutely arbitrary we should have particular cases of sextics with 5 or 4 cusps, and genus one. By modifying AB and AC we should obtain the remaining cases.

(3) *Genus superior to one.* The existence of all sextics with positive characters, for $p > 1$, follows from the proposition given in § 12, and if necessary their existence could be shown by the methods used for rational and elliptic sextics, but we will not insist.

Existence of the curves of genus 1, 2, 3, 4 and 5.

16. *Rational curves.* We have given (§ 4) the equation of these curves when $\kappa \leqq m - 2$. We shall now show *the existence of all rational curves with less than* $[\frac{3}{2}(m-2)]$ *cusps.* Suppose that

$$\kappa = (m-2) + g, \qquad g \leqq 1(m-2).$$

Then we find:

$$n = 2(m-1) - \kappa = m - g > 0,$$

$$i = 3(m-2) - 2(m-2+g) = m - 2 - 2g$$

$$= m - g - (g+2) = n - g - 2 < n - 2.$$

If we consider then the equation:

$$z(x - ay)^{m-1-2g}\varphi_g(x, y) + \varphi_{m-g}(x, y) = 0,$$

it represents a curve the reciprocal of which has $m - 2 + g$ cusps, hence

$$w(u - av)^{m-1-2g}\varphi_g(u, v) + \varphi_{m-g}(u, v) = 0$$

is the tangential equation of a rational $m$thic with $(m - 2 + g)$ cusps $(g \leqq [\frac{1}{2}(m-2)])$.

It is interesting here to recall Veronese's * method of establishing the same result. More generally he showed that the curve having the characters (m, p, κ) exists if $\kappa \leqq \frac{3}{2}[m - p - 2]$. If $p = 0$ we obtain $\frac{3}{2}[m - 2]$. He considers the C_m^{m-p} of order m and genus p, in $(m - p)$ fold space, and shows that there can always be found an $(m - p - 3)$fold flat meeting $\frac{3}{2}[m - p - 2]$ (or less) straight lines, in particular tangents of C_m^{m-p}, in the space considered. By projecting then from the $(m - p - 3)$ flat into a plane, a C_m^2 is obtained with $\frac{3}{2}(m - p - 2)$ cusps or less. There is little doubt that by extending the method of § 14 as indicated at the end, we could obtain equations or curves with any number of cusps up to $2m - 7$ and assigned

* Loc. cit.

deficiency — within certain limits perhaps — and this number of cusps is much higher than the limit given by Veronese. It may be stated that Veronese's proof shows that *any curve of order m and genus p can be birationally transformed into another of the same order in such a way that* $\frac{3}{2}(m-p-2)$ *of its points become cusps of the transformed curve.*

In his memoir Veronese states also that the extension of his method to the case $p \neq 0$, depends upon an investigation of the number of tangents of a C_m^{m-p} in an $(m-p)$fold space that can meet an $(m-p-3)$-flat of this space. Our problem is then so to dispose of the moduli (in Riemann's sense) of C_m^{m-p} that it have a maximum number of tangents meeting an $(m-p-3)$-flat, and in this disposing of the moduli lies the difficulty.

Elliptic curves. The lowest value of m for which the theorem of § 13 applies in this case is: $m = 2.1 + 1 + \sqrt{8+1} = 6$ and by § 15 follows the certainty of the existence of all elliptic curves with positive or zero characters.

17. *Curves of genus two.* From § 13 follows their existence when

$$m > 2.2 + 1 + \sqrt{8.2+1} = 9, 1 \text{ or } 10.$$

For $m = 7, 8, 9$, a direct examination is necessary.

Case 1. $m = 7$. We must have $\kappa \leqq \kappa_1$ and we find

$$\kappa_1 = [\tfrac{1}{2}(4(7+2) - 11 - \sqrt{-8.7 + 24.2 + 25})] = 10.$$

But for $m = 7, p = 1$, the limit is $\kappa \leqq [\frac{3}{2}[7 + 2 - 2]] = 10$. So that we know that all curves for which $m = 7$, $p = 1$, $\kappa \leqq 10$, exist, hence (§ 12) the same is true for the curves such that $m = 7$, $p = 2$, $\kappa \leqq 10$.

Case 2. $m = 8$. We find here $\kappa_1 = 13$, so that we must have $\kappa \leqq 13$. The limit for the case $m = 8$, $p = 1$ is $[\frac{3}{2}(8 + 2. - 2)] = 12$. Hence the only curve to be considered is the one for which $\kappa = 13$. But its reciprocal is found to be a quintic, and therefore exists necessarily, so that the same is true for the curve in question.

Case 3. $m = 9$. In this case $\kappa_1 = 16$. For $p = 1$, $m = 9$ it is found that $\kappa \leqq 13$, while if $\kappa = 14, 15, 16$, the curve is the reciprocal of curves of order $6, 5, 4$, hence its existence is certain.

We have therefore shown the existence of all curves of genus two with positive characters.

18. *Curves of genus three.* From § 13 follows their existence when $m \geq 2.3 + 1 + \sqrt{8.3+1} = 12$. We must therefore consider separately curves of order $7, 8, 9, 10, 11$.

(1) $m = 7$. In this case $\kappa_1 = 11$. When $p = 2$ we know that κ can have any value inferior to 10, and we must therefore merely prove the existence of the curve for which $\kappa = 11$. The characters of the reciprocal curve are the same, and none of the other methods of this paper succeed with this curve, the existence of which we have not succeeded in proving as yet.

(2) $m = 8$. Here $\kappa_1 = 13$, the same limit as for $p = 2$, hence by § 12 we know that κ can have all values smaller than 13.

(3) $m = 9$. Here $\kappa_1 = 16$, and the limit is again the same as for $p = 2$.

(4) $m = 10$. Again $\kappa_1 = 18 = \beta(10, 2)$.

(5) $m = 11$. Here $\kappa_1 = 21$, and for $p = 2$ we must have $\kappa \leqq \beta(11, 2) = 19$. Hence we have to consider only the cases $\kappa = 20, 21$; but then the class being less than 9, we are reduced to a preceding case. *We can therefore conclude to the existence of all curves of genus three with positive characters, the curve* (7, 3, 11) *being excepted.*

19. Without further detailed discussion, we shall merely state that this method can be used to prove the existence of all curves of genus four or five, with the exception of the following two:

$$m = 8, \qquad p = 4, \qquad \kappa = 14;$$

$$m = 8, \qquad p = 5, \qquad \kappa = 16;$$

as they have the class equal to the order. Moreover it can be easily shown that when m and p both increase there is always a value of m limiting the range of application of the method for a given value of p. More precisely, if

$$m \leqslant \frac{\sqrt{24p - 39} + 9}{2} \qquad (p > 3),$$

some of the curves whose existence is to be proved will have reciprocals of order higher than their own, or equal to it.

Singularities of surfaces and higher varieties.

20. Surfaces and varieties of $(n-1)$ dimensions in n-space can be treated like plane curves. It can be shown for them also that they have in general $\frac{m(m-1)\cdots(m-n+1)}{n!} - (n+1)^2$ absolute invariants, unless they are invariant under a continuous projective group of transformations. Hence * by a well-known theorem: *When a surface of order m has a positive numerical genus p_n, it has exactly*

$$\frac{(m+1)(m+2)(m+3)}{6} - 16$$

absolute invariants. For a surface invariant under any group of transformations can be birationally transformed into a surface, for which $p_n \leqq 0$. In particular a surface of order greater than 3 and having no other singularities than nodes or binodes satisfies the above condition. Another criterion

* Picard et Simart: *Traité des fonctions algebriques de deux variables*, vol. 2, page 520.

which we shall not develop here but which results easily from a short paper of Aronhold * states that: *If the first polars of m independent points have in common a system of* $(m+1)$ *points not situated on an arbitrary first polar and forming a true* $(m+1)$*-edron, then the variety is not invariant under any projective group and the number of absolute invariants is as given above.*

To find the *a priori* possible surfaces or varieties we discuss the tangent hypercone. In case of a surface, the section of the tangent cone will have a minimum number of double points and cusps, and the question will be reduced to the one outlined in § 11. If δ is the number of quadratic nodes, κ the number of ordinary quadratic binodes, and if the surface has no other singular points or curves than these, then we have the condition

$$\delta + 2\kappa \leqq \frac{(m+1)(m+2)(m+3)}{6} - 16 + \lambda$$

with $\lambda = 0$ if the *extended postulate of singularities* is admitted.

21. Let us outline briefly the discussion for *quartic surfaces.* In this case there are $5.6.7/6 - 16 = 19$ absolute invariants. The tangent cone is of order 12, and has $\frac{1}{2}.4.3.2.1. + \delta = 12 + \delta$ double edges and $4.3.2 + \kappa = 24 + \kappa$ cuspidal edges.

Let n be the class of this cone, i and τ the number of its inflexional and bitangential planes. We then have:

$$n = 12.11 - 2(\delta + 12) - 3(\kappa + 24),$$

$$i = 3.12.10 - 6(\delta + 12) - 8(\kappa + 24),$$

$$2\tau = 2(\delta + 12) + (n - 12)(n + 12 - 9),$$

$$\delta + 2\kappa \leqq 19.$$

If instead of p, we introduce $\delta + \kappa = \lambda$ and write the conditions that the characters be positive or zero, we obtain:

$$36 - 2\lambda - \kappa \geq 0,$$

$$48 - 3\lambda - \kappa \geq 0 \qquad \lambda \infty 18,$$

$$\lambda + \kappa \leqq 19,$$

$$2\tau = (\kappa + 2\lambda)^2 - 124\kappa - 65\lambda + 960 \geq 0;$$

and the discussion is best carried out by means of diagrams, which we will omit here. The following results worthy of notice are then obtained:

(1) *The number of nodes cannot exceed* 16 — *a well-known fact.*

(2) *If the number of nodes is inferior to* 8 *they can all be binodes.*

* *Fundamentalsatz der Invariantentheorie.* Crelle's Journal, vol. 69 (1868), p. 185.

(3) *There cannot be any binodes when the total number of nodes exceeds* 14.

It is found by a similar discussion that a quintic surface cannot have more than 34 nodes, and that if it has one binode the number of nodes is not superior to 33, while the total number of binodes can never exceed 20.

22. This discussion applied to cubic varieties gives several interesting results. In the first place the maximum number of nodes as shown by the postulate of singularities, viz. 10, can be reached, and we have then the Segre variety. Next, this postulate gives 5 as the maximum number of nodes of the second species, and the writer has shown elsewhere the existence of the corresponding variety.*

In conclusion our thanks are due to Professor W. E. Story for suggesting the investigations of this paper.

WORCESTER, MASS.,
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*Bulletin of the American Mathematical Society, vol. 18 (1912), p. 384.

SINGULAR MULTIPLE INTEGRALS, WITH APPLICATIONS TO SERIES*

BY

BURTON H. CAMP

Introduction.

The object of this paper is the study of the convergence to zero of Lebesgue integrals of the type

$$\int_E f(t, u, \cdots)\, \phi(t, u, \cdots; n, m, \cdots)\, dE,$$

and to $f(x, y, \cdots)$ of singular Lebesgue integrals of the type

$$\int_E f(t, u, \cdots)\, \theta(t, u, \cdots; x, y, \cdots; n, m, \cdots)\, dE,$$

where E is a limited, measurable field in space of i dimensions, f is a function of i variables, $(x, y, \cdots)$ is a fixed point in E, and $n, m, \cdots$ are j parameters that grow infinite simultaneously but independently.† In a recent memoir LEBESGUE ‡ has considered the same problems for the case where i and j are unity. The main part of the present discussion consists in the extension of his theorems so that they will apply to the more general integrals. The proofs of these theorems are long, and accordingly, in those cases where important new difficulties are not encountered by the addition of more variables, and also in certain other relatively unimportant instances, the proofs are omitted.

These integrals arise, in practice, especially in connection with the development of functions in series of normal functions, where the coefficients have the

* Presented to the Society, April 27, 1912.

† I. e., $n, m, \cdots$ belong to sequences which are independent except that they diverge to plus infinity at the same rate. This restriction concerning the rate is essential only to the proofs that the conditions are *necessary*. In all the convergence theorems of this paper, then, the conditions remain *sufficient* without this restriction, provided only it be understood that everywhere multiple, not iterated, limits are meant.

‡ Annales de la Faculté de Toulouse, ser. 3, vol. 1 (1909), pp. 25–128. Cf. also two theorems of HOBSON in which the sufficient conditions of two of LEBESGUE's theorems are extended so as to apply to the more general integrals, Proceedings of the London Mathematical Society, ser. 2, vol. 6 (1908), pp. 349 ff.

form of the first integral, and the sum of the first $n\,m\,\cdots$ terms can be put in the form of the second. As applications of the theory, therefore, I have given necessary and sufficient conditions that the coefficients of such a series converge to zero, and have proved a theorem concerning the development of a function in double Fourier series. Finally, I have indicated some applications which may be made to the evaluation of multiple integrals by means of series. As far as I know, the best recent work on the subject of double Fourier series has been done by HARDY * (1906). Other articles that have come to my notice are those written by ASCOLI † (1880), by PICARD ‡ (1901), by CERNI § (1901), and by VERGERIO ‖ (1911). Theorems corresponding to mine on the evaluation of definite integrals, for the case where the integrands are functions of a single variable, have been given by W. H. YOUNG. ¶ In a paper on *Sturm-Liouville series of Normal Functions in the Theory of Integral Equations*, MERCER ** has had occasion (page 124) to prove for one variable the sufficiency of 1° in my eighth theorem. Of his two fundamental convergence theorems, the first may readily be deduced from the sufficient part of my Theorem 1, and the second has no close connection with any of my theorems.

§ 1. Conditions that the integral

$$\int_A f(t, u, \cdots)\,\phi(t, u, \cdots; n, m, \cdots)\,dA,$$

where A is any limited, measurable field in space of i dimensions, f and ϕ are defined in A, and $n, m, \cdots$ are j parameters that grow infinite simultaneously but independently,†† may exist, for sufficiently large values of $n, m, \cdots$, as an absolutely convergent L-integral ‡‡ and approach zero for all functions f, of i variables, belonging to one of the families specified below, may be stated in the several cases as follows:

THEOREM 1: *If F is the family F_1 of absolutely L-integrable §§ functions, it is*

* Quarterly Journal of Mathematics, vol. 37 (1906), p. 53.

† Atti della Reale Accademia dei Lincei, Memorie, ser. 3, vol. 8 (1880), pp. 263–319.

‡ Traité d'Analyse, 2d ed. (1901), p. 294.

§ Reale Istituto Lombardo, Rendiconti, ser. 2, vol. 34 (1901), p. 921.

‖ Giornale di Matematiche, vol. 49 (1911), pp. 181–206.

¶ Proceedings of the London Mathematical Society, ser. 2, vol. 9 (1910), p. 464.

** Transactions of the Royal Society of London, *A*, vol. 211 (1911), pp. 111 ff.

†† After obvious verbal changes, these theorems are valid also in case the parameters converge to a fixed point $(n_0, m_0, \cdots)$ simultaneously but independently. See also the first footnote to this paper.

‡‡ I. e., integral in the sense of LEBESGUE.

§§ It should be noted that the conditions automatically require ϕ to be absolutely L-integrable, when the parameters are large, except in Theorem 6, where this requirement is separately stated.

necessary and sufficient that (1°) *M exist so that* $|\phi|$ *be less than M when* * $n, m, \cdots > n_M$, *except perhaps at a null set of points; and that either* (2°) *the integral of* ϕ *over each measurable set a in A, or else* (2a°) *the integral of* ϕ *over each standard cube*† q, *approch zero as n becomes infinite.*

COROLLARY: *If F is the family* F_1' *of L-integrable functions having only a finite number of values in A, it is necessary and sufficient that* 2° *of the theorem be satisfied.*

THEOREM 2: *If F is the family* F_2 *of functions whose squares are L-integrable in A, it is necessary and sufficient that* (1°) *M exist so that the integral over A of* ϕ^2 *be less than M, when* $n, m, \cdots > n_M$, *and that either* 2° *or else* 2a° *of Theorem* 1 *be satisfied.*

THEOREM 3: *If F is the family* F_3 *of functions which are limited and L-integrable in A, it is necessary and sufficient that* (1°) *M exist so that the integral over A of the absolute value of* ϕ *be less than M when* $n, m, \cdots > n_M$, *and that* 2° *of Theorem* 1 *be satisfied; it is also necessary and sufficient that* (1a°) *there exist for an arbitrary* $\gamma > 0$ *two numbers,* $\lambda_\gamma > 0$, *and* n_γ, *so that for every measurable set b in A whose measure is less than* λ_γ, *the integral over b of the absolute value of* ϕ *is less than* γ *when* $n, m, \cdots > n_\gamma$, *and that* 2a° *of Theorem* 1 *be satisfied.*

COROLLARY 1: *The meaning of condition* 1a° *is not changed if either of the following substitutions be made.* (1) *The integral of the absolute value of* ϕ *may be replaced by the absolute value of the integral,*‡ *or* (2) *b may be replaced by an infinite set of non-overlapping,*§ *standard cubes* $\{q_i\}$.

COROLLARY 2: *For the truth of the conclusion,* 1a° *and* 2° *are likewise necessary and sufficient; but* 1° *and* 2a° *are necessary and not sufficient.*

COROLLARY 3: *If* ϕ *does not change sign in each of a finite number of measurable sets whose sum is A,* 2° *is necessary and sufficient.*

THEOREM 4: *If F is the family* F_4 *of "simply discontinuous functions" (defined below* § 3) *in the limited, dense, metric* ‖ *set C, it is necessary and sufficient that* (1°) *M exist so that the integral over C of the absolute value of* ϕ *be less than M when* $n, m, \cdots > n_M$, *and that* (2°) *the integral of* ϕ *over each dense, metric set c in C approach zero as n becomes infinite.*

COROLLARY: *If F is the family* F_4', *a partial family of* F_4 *defined by means of a particular, previously chosen division of* C: $C_1 + \cdots + C_\mu$ (*see* § 3), *it is sufficient*

* I use n_M in place of the customary n_0 to denote a fixed number. In the proof of the necessary part of this theorem I need to use a sequence of M's and a corresponding sequence of n_M's. Similar reasons hold for analogous notations in other theorems.

† I. e., a cube in space of i dimensions whose edges are parallel to the coördinate axes. Since ϕ is defined only in A, the integral is taken over only those points of q that are in A.

‡ In this form the condition is that the integral of ϕ over A be "equi-absolutely continuous" in the sense of VITALI (Rendiconti di Palermo, vol. 23 (1907), p. 139).

§ Except perhaps in their boundary surfaces.

‖ I. e., having content in the sense of CANTOR, cf. PIERPONT, *Theory of Functions of Real Variables*, vol. 2, p. 1.

that 1° *hold, and that* (2a°) *for each* C_i 2a° *of Theorem* 1 *be satisfied, the integral being taken over only that part of* q *which is in* C_i $(i = 1, \cdots, \mu)$

THEOREM 5: *If* F *is the family* F_5 *of continuous functions of two variables in the region* D *(defined below,* § 4*), it is sufficient that* (1°) M *exist so that the integral over* D *of the absolute value of* ϕ *be less than* M *when* $n, m, \cdots > n_M$, *and that* (2°) *the integral over* D *of* ϕ *approach zero as* n *becomes infinite, and that* (3°) *for each* e *the integral over* e *of* $(t \cos \alpha + u \sin \alpha - p)\ \phi$ *approach zero as* n *becomes infinite, where* e, α, *and* p *are as defined below* * (§4). *It is necessary that the same conditions be satisfied, except that the region* e *is to be replaced by the region* d *(defined below,* § 4).

THEOREM 6: *If* F *is the family* F_6 *of functions "monotone increasing with respect to the point sets* $\{B_\lambda\}$" *(defined below,* § 5), *it is sufficient that* ϕ *be absolutely L-integrable, and that* (1°) *the integral of* ϕ *over* B_λ *approach zero uniformly for all* λ*'s as* n *becomes infinite. It is necessary that* (2°) M *exist so that, if* $n, m, \cdots > n_M$, ϕ *is absolutely L-integrable and the absolute value of its integral over* B_λ *is less than* M *for all* λ*'s uniformly, and that* (3°) *for each value of* λ *this integral approach zero as* n *becomes infinite.*

COROLLARY 1: *If* F *is the family* F_6' *of functions of two variables of limited variation in the sense of Hardy* †, *it is sufficient that* 1° *be satisfied uniformly for all* B_λ*'s, when* B_λ *and* A *are "simple domains containing the origin, and the coördinates of each point of* A *are positive or zero."* ‡

COROLLARY 2: *If* F *is the family* F_6'' *of functions of two variables having limited variation in the sense of* PIERPONT § *the conditions* 2° *and* 3°, *with reference to each set of "monotone increasing sets* $\{B_\lambda\}$," *are necessary,* B_λ *and* A *being as in Corollary* 1.

It is to be noted that if a function of two variables has limited variation in the sense of Hardy it may be expressed as the difference of two functions monotone increasing ‖ with respect to the coördinates, but that not all functions which may be so expressed have limited variation. On the other hand, not all functions that have limited variation in the sense of Pierpont may be expressed as the difference of two such functions, but all functions that are so expressible do have limited variation. *Example* 1. The function which equals zero in that half of the square $(0 \leqq x, y \leqq 1)$ that lies between the origin and the line, $x + y = 1$, and equals unity in the remainder, is monotone in-

* These conditions are less severe than those of Theorem 4, even when C is replaced by D.

† HARDY. Quarterly Journal of Mathematics, loc. cit., p. 58. Cf. also LEBESGUE, Annales de l'École Normale, ser. 3, vol. 27 (1910), p. 408.

‡ In the paper mentioned in the preceding note (p. 445) LEBESGUE defines such a domain as one having the property that, if the point P belongs to it, so does the rectangle formed by the coördinate axes and perpendiculars to them through P.

§ PIERPONT, loc. cit., vol. 1 (1903), p. 518.

‖ Cf. HARDY, loc. cit., p. 58 (14), and p. 59 (*i*). Functions of limited variation, as HARDY defines them, are limited and are expressible as the difference of two limited monotone increasing functions.

creasing and has limited variation in the sense of Pierpont, but not in the sense of Hardy. *Example* 2. The function which is equal to unity in the circle, $x^2 + y^2 = 1$, and to zero elsewhere in a square enclosing this circle is not the difference of two limited monotone increasing functions, and consequently does not have limited variation in the sense of Hardy, but it does have limited variation in the sense of Pierpont.

PROOFS OF THE PRECEDING THEOREMS.

§2. *Proofs of Theorems* 1 *and* 2.—The proofs of these theorems for the conditions 1° and 2° are similar to those given by Lebesgue for the one-dimensional case. The substitution of 2° for Lebesgue's $2a°$ simplifies the work, especially if the corollary of Theorem 1 be established first. To show the necessity and sufficiency of the other conditions the following lemmas will be used.

LEMMA* 1: *If each of two functions is absolutely L-integrable in the limited, measurable field* A, *and one of them is limited, except perhaps in a null set, their product is also absolutely L-integrable in* A.

LEMMA 2: *If* $1a°$ *of Theorem* 3 *is satisfied, then* 1° *is also satisfied.*

Suppose $\gamma = 1$. By hypothesis $\lambda_1 > 0$ and therefore† $\bar{A}/\lambda_1 < \infty$. Set $M = 2\bar{A}/\lambda_1$, and let $A = A_1 + \cdots + A_p$, where $\frac{1}{2}\lambda_1 < \bar{A}_i < \lambda_1$, $i = 1, 2, \cdots, p$. Then $\bar{A} > \frac{1}{2}p\lambda_1$, or $p < 2\bar{A}/\lambda_1 = M$. By hypothesis there exists an n_1 so that

$$\int_{A_i} |\phi| < 1 \quad (n, m, \cdots > n_1,\ i = 1, \cdots, p).$$

$$\therefore \int_A |\phi| = \sum_1^p \int_{A_i} |\phi| < p < M \quad (n, m, \cdots > n_1).$$

LEMMA 3: *If* $1a°$ *and* $2a°$ *of Theorem* 3 *are satisfied, then* 1° *and* 2° *are satisfied, but* 1° *and* $2a°$ *are not sufficient to ensure* 2°.

By the preceding lemma, 1° is satisfied if $1a°$ is. To establish 2°, let γ be greater than zero and arbitrary, and let a be an arbitrary, measurable set in A, and enclose a in a set of non-overlapping, standard cubes, $q_1, q_2, \cdots$, and suppose p, depending on λ_γ and a, to be so chosen that

$$\bar{q}_1 + \cdots + \bar{q}_p + \bar{q}_{p+1} + \cdots - \bar{a} < \lambda_\gamma, \tag{1}$$

and

$$\bar{q}_{p+1} + \cdots < \lambda_\gamma. \tag{2}$$

*Annales de l'École Normale, loc. cit., p. 374, § 13. In the beginning of this article will be found the principal properties of "measure" and of multiple *L*-integrals used in the present paper. For a more complete treatment the reader should consult PIERPONT, loc. cit., vol. 2.

† Let "measure of u" be denoted by $\bar{u}$. See preceding note.

Then, by (2) and $1a°$, and by (1) and $1a°$,

$$(3) \qquad \left| \int_{\sum_1^\infty q_i} \phi - \int_{\sum_1^p q_i} \phi \right| = \left| \int_{\sum_{p+1}^\infty q_i} \phi \right| < \gamma \qquad (n, m, \cdots > n_\gamma),$$

$$(4) \qquad \left| \int_a \phi - \int_{\sum_1^\infty q_i} \phi \right| = \left| \int_{\sum_1^\infty q_i - a} \phi \right| < \gamma \qquad (n, m, \cdots > n_\gamma).$$

Adding (3) and (4), and using $2a°$, viz.,*

$$\left| \int_{\sum_1^p q_i} \phi \right| < \gamma \qquad (n, m, \cdots > n_{p,\gamma} = N_{\gamma,a}),$$

we have

$$\left| \int_a \phi \right| < 3\gamma \qquad (n, m, \cdots > n_\gamma, N_{\gamma,a}).$$

An example of a function satisfying $1°$ and $2a°$ but not $2°$ is given below.†

LEMMA 4: *If* $1°$ *of Theorem* 2 *is satisfied, so is* $1a°$ *of Theorem* 3.

For, for any $\gamma > 0$ we may select $\lambda_\gamma > 0$ so that $\sqrt{\lambda_\gamma M} < \gamma$, and then, if $\bar{b} < \lambda_\gamma$ and b is in A, by the inequality of Schwarz,‡

$$\left| \int_b \phi \right| < \sqrt{\int_b \phi^2 \int_b 1} < \sqrt{M\bar{b}} < \sqrt{M\lambda_\gamma} < \gamma. \qquad (n, m, \cdots > n_\gamma).$$

Now, in Theorem 2, the necessity of $2a°$ follows from the necessity of $2°$. $1°$ and $2a°$ are sufficient because, from them, by Lemmas 4 and 3, $2°$ follows, and $1°$ and $2°$ are already known to be sufficient.

Similar considerations complete the proof of Theorem 1, for evidently, if $1°$ of Theorem 1 is satisfied, so is $1°$ of Theorem 2.

Proof of Theorem 3.—Lebesgue's method suffices to show that $1a°$, $2°$, and $2a°$ are necessary. From Lemma 2 it then follows that $1°$ is necessary. By Lemma 3, then, all that remains to show in order to establish the theorem is that $1°$ and $2°$ are sufficient.

Let $-F < f < F$, and let the interval $(-F, F)$ be divided by the points, $a_0 = -F, a_1, a_2, \cdots, a_p = F$, so that $a_{i+1} - a_i < \epsilon / 2M$, $i = 0, \cdots, (p-1)$. For each of these values of i let e_i be the set of points in A where $a_i \overline{\leq} f < a_{i+1}$, and let g be the function which equals a_i in each e_i. Then, by Lemma 1,

$$(1) \qquad \left| \int_A f\phi - \int_A g\phi \right| = \left| \int_A (f - g)\,\phi \right| < \frac{\epsilon}{2M} \int_A |\phi| < \frac{\epsilon}{2},$$

* $n_{p,\gamma}$ is a fixed number after p and γ are chosen, and p depends only on γ and a.

† In the proof of Corollary 2.

‡ Annales de l'École Normale, loc. cit., p. 442.

if $n, m, \cdots > n_M$. But, by the corollary to Theorem 1, and by 2°,

$$\left|\int_A g\phi\right| < \frac{\epsilon}{2}. \qquad (n, m, \cdots > n_{p,\epsilon}). \tag{2}$$

By the addition of (1) and (2) the theorem is now established.

Proof of Corollary 1 (*Theorem* 3).—Since the absolute value of the integral of a function is at most equal to the integral of its absolute value, it is evident that all that requires proof in (1) is that, assuming that for each $\gamma > 0$ there exist $\lambda_\gamma > 0$ and n_γ so that the absolute value of the integral over b of ϕ is less than γ if $\bar{b} < \lambda_\gamma$, and $n, m, \cdots > n_\gamma$, then $1a°$ follows.

To prove this, having fixed $n, m, \cdots > n_\gamma$, we may divide b into two parts b' and b'', so that ϕ is nowhere negative in b', and everywhere negative in b''. Both b' and b'' are now special cases of b.

(2) is readily established, since, if $\bar{b} < \frac{1}{2}\lambda_\gamma$, there exists a set of non-overlapping,* standard cubes $\{q_i\}$, enclosing b, so that

$$\bar{b} \leqq \frac{\lambda_\gamma}{2} \leqq \sum_1^\infty \bar{q}_i < \lambda_\gamma.$$

Proof of Corollary 2 (*Theorem* 3).—The following function satisfies 1° and $2a°$, and not 2°.

Let $A = (0, 1)$, $\{n\} = 1, 2, 2^2, \cdots$. Divide the interval $(0, 1)$ by the points $0, 1, \frac{1}{2}, (\frac{1}{2})^2, (\frac{1}{2})^3, \cdots$, and let $\delta_i = [(\frac{1}{2})^{i+1}, (\frac{1}{2})^i]$, $i = 0, 1, \cdots$. Divide each δ_i into two equal intervals, δ_i' and δ_i''. Let

$$\begin{aligned} \phi(t, n) &= n \text{ in each } \delta_i', \ 0 \leqq t \leqq 1/n, \\ &= -n \text{ in each } \delta_i'', \ 0 \leqq t \leqq 1/n, \\ &= 0 \text{ at } t = 0, \text{ and where } t > 1/n. \end{aligned}$$

This is the function desired. It shows that 1° and $2a°$ are not sufficient. The rest of the corollary follows from Lemma 2 and the theorem.

§ 3. *Explanation and Proof of Theorem* 4.—We shall say that f is "uniformly continuous" in a limited, metric,† dense set C, if it is defined in C, and for each $\epsilon > 0$ there exists a finite set of non-overlapping,‡ standard cubes, $q_1, q_2, \cdots, q_p$, enclosing C, so that § in each q os$f < \epsilon$. It is known that this condition is satisfied if, in particular, C is perfect, and f is continuous in C.

* Except perhaps in their bounding surfaces.

† See footnote to Theorem 4, § 1.

‡ Except perhaps in their bounding surfaces.

§ "Osf" is an abbreviation for "oscillation of f," *i. e.*, $\max f - \min f$. I use the terms maximum and minimum in the sense of upper and lower limits, not of greatest and least values. In the proof, after the first clause and until equation (4) is reached, f refers only to those values of f in C_4.

We shall say that f is "simply discontinuous" in the same set C if it is defined in C and there exists a division of C into a finite number of metric, dense parts so that in each part f is uniformly continuous.

Sufficiency.—For a chosen f, let C_i be one of these metric, dense parts of C, and let C_i be enclosed in $q_1, \cdots, q_p$, so that in q_j

$$\text{os} f < \epsilon \qquad (j = 1, \cdots p). \tag{1}$$

Let $g = k_j = \min f$ in q_j $(j = 1, \cdots, p)$. Then by 2°, the integral over q_j having reference only to that part of q_j which is in C_i,

$$\int_{C_i} g\phi = \int_{q_1} g\phi + \cdots + \int_{q_p} g\phi = k_1 \int_{q_1} \phi + \cdots + k_p \int_{q_p} \phi, \tag{2}$$

and approaches zero as n becomes infinite. By (1) and 1°,

$$\left| \int_{C_i} f\phi - \int_{C_i} g\phi \right| = \left| \int_{C_i} (f - g)\,\phi \right| \leqq \epsilon \int_{C_i} |\phi| < \epsilon M \qquad (n, m, \cdots > n_M). \tag{3}$$

From (2)

$$\left| \int_{C_i} g\phi \right| < \epsilon, \qquad (n, m, \cdots > n_\epsilon)$$

and this added to (3) shows that the integral over C_i of $f\phi$ approaches zero as n becomes infinite. Hence, since the number of C's is finite, and since

$$\Sigma \int_{C_i} f\phi = \int_C f\phi, \tag{4}$$

the last integral also approaches zero.

It is apparent that this proof applies to the corollary.

Necessity.—2° is evidently necessary, for, if a c exists for which 2° is not satisfied, the theorem is contradicted by the function which equals unity in c and zero elsewhere.

I will now show that if 1° is not satisfied there exists a continuous function, which also vanishes on the boundary of C, contradicting the theorem.

LEMMA 1: *Let Q be a finite set of non-overlapping,* standard cubes, and let $\psi(t, u, \cdots)$ be defined in Q, and, for some $h > 0$, let the integral over Q of the absolute value of ψ be greater than h. Then there exists in Q a continuous function, $p(t, u, \cdots)$, which is numerically $\leqq 1$, and which vanishes on the boundary of Q, such that the integral over Q of $p\psi$ is greater than h.*

This may be shown by a method used by Lebesgue in his proof of the theorem corresponding to Theorem † 4.

* Except perhaps in their bounding surfaces.

† Loc. cit., p. 61.

LEMMA 2: *If, in the hypothesis of Lemma* 1, *C be substituted for Q, then there exists in C a continuous function g, vanishing on the boundary of C, so that the integral over C of* $g\psi$ *is greater than h, and* $|g| \leqq 1$.

By hypothesis there exists a $\gamma > 0$ so that

$$\int_C |\psi| > h + \gamma. \tag{1}$$

Since by hypothesis C cannot have content zero, for an arbitrary $\zeta > 0$ there exists a Q set of cubes, all points of which are in C, so that

$$0 < \bar{C} - \bar{Q} < \zeta. \tag{2}$$

Choosing $\zeta > 0$ so that, by a known theorem,*

$$\int_b |\psi| < \gamma, \quad \text{if} \quad \bar{b} < \zeta, \tag{3}$$

we have, by (2) and (3),

$$0 \leqq \int_C |\psi| - \int_Q |\psi| = \int_{C-Q} |\psi| < \gamma.$$

Hence, by (1),

$$\int_Q |\psi| > h.$$

Now, using Lemma 1, let g be equal to p in Q and to zero in the rest of C. Then

$$\int_C g\psi = \int_Q p\psi > h,$$

and, since p vanishes on the boundary of Q, g satisfies the requirements of the lemma.

We continue now the proof of the theorem. If 1° is not satisfied; for each N_0, and each $h_0 > 0$, there exist $n_0, m_0, \cdots > N_0$ so that

$$\int_C |\phi(n_0, m_0, \cdots)| > h_0.$$

Then by Lemma 2 there exists in C a continuous function g_0, vanishing on the boundary of C, and numerically $\leqq 1$, so that

$$\int_C g_0 \phi(n_0, m_0, \cdots) > h_0.$$

We may now proceed without difficulty in the manner outlined by Lebesgue.†

*Annales de l'École Normale, loc. cit., p. 374.

†Annales de la Faculté de Toulouse, loc. cit., p. 61.

§ 4. *Explanation and Proof of Theorem* 5.—We make the following definitions. D is a complete, metric * region in the tu-plane containing a point O such that every straight line through O cuts the closed curve forming the boundary of D in exactly two points. There shall exist a fixed number which is greater than the number of intersections of the boundary of D with any straight line. That portion of D which lies on the opposite side from O of any straight line is denoted by d; it includes its boundary; if the line passes through O, each portion is denoted by d. The equation of this striaght line shall be: $t\cos\alpha + u\sin\alpha = p$. That portion of d which lies between two radial lines, including at O an acute angle, is denoted by e; e also includes its boundary.

We begin by showing that the surface f may be approximated by a continuous surface F made up of plane facets. Let $\epsilon > 0$ and $\delta > 0$ be such that, in any region which can be enclosed in a circle of diameter less than δ, $osf < \epsilon$. Let D be enclosed in a regular polygon Q, whose

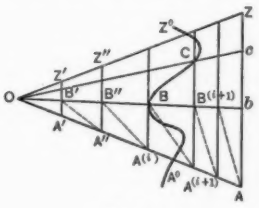

FIG. 1.

center is O and whose sides have a length $< \frac{1}{2}\delta$, and let Q be subdivided in the following manner. AZ being any side, triangle AZO is partitioned by parallel lines, $A'Z'$, $A''Z''$, $\cdots$, AZ, into parts so small that each can be enclosed in a circle of diameter less than δ. Each of the finite number of sides of all these parts is cut by the boundary of D at most a finite number of times, so that there are between OA and OZ a finite number of points of intersection, A°, B, C, $\cdots$, Z°; A° and Z° lying on OA and OZ respectively, and perhaps coinciding with some points already named. Draw the lines OBb, OCc, $\cdots$; b, c, $\cdots$ being in AZ; and fix the attention on S_1, the part of D in AOb. Let B', B'', $\cdots$ be the intersections of Ob and $A'Z'$, $A''Z''$, $\cdots$. For some i, $OA^{(i)} \leqq OA^\circ \leqq OA^{(i+1)}$. By construction, as one goes along the boundary of D from A°, the next point of intersection B with the network is on Ob and also on a line parallel to AZ. This point of intersection will therefore coincide with $B^{(i-1)}$, $B^{(i)}$, $B^{(i+1)}$, or $B^{(i+2)}$. Without loss of generality, as will become apparent, we may suppose that $B^{(i)} = B$, and that D contains $A^{(i)}OB$. Draw $A''B'$, $A'''B''$, $\cdots$, $A^{(i+1)}B^{(i)}$. Let τ_1 be $OA'B'$, τ_2 be $A'A''B'$, $\cdots$, τ_κ be the part of S_1 in $A^{(i+1)}A^{(i)}B^{(i)}$, and τ_l the part in

* See note to Theorem 4, § 1.

$A^{(i+1)}\,B^{(i)}\,B^{(i+1)}$. The essential characteristics of each τ are that in it

$$\text{(1)} \qquad \text{os}f < \epsilon,$$

and, except in the case of τ_1, and perhaps of τ_κ, the three vertices are points of D. Let $y\,(B^{(i+1)})$ be a point whose projection on the tu-plane is $B^{(i+1)}$ and whose distance from it equals the distance of $f\,(B^{(i)})$. If $A^{(i+1)}$ does not belong to D, let $y\,(A^{(i+1)})$ be a point whose projection is $A^{(i+1)}$ and whose distance equals the distance of $f\,(A^{(i)})$. Through the points of the f surface, $f(O), f(A'), f(B')$, pass a plane P_1, and let the part whose projection is τ_1 be π_1. Proceeding in a similar manner with the other τ's (using the y's just defined at vertices where f is not defined), and continuing the process over all the other sectors, $S_2, S_3, \cdots$, we shall have finally the surface F of facets desired; for, since the distance of every other point of any facet from the tu-plane lies between the greatest and least distances of its vertices, by (1)

$$\text{(2)} \qquad |F - f| < \epsilon.$$

We will now show that

$$\text{(3)} \qquad F = c + \sum_1^r f_i,$$

where $f_i = a_i\,(t\cos\alpha_i + u\sin\alpha_i - p_i)$ in e_i, $f_i = 0$ in $D - e_i$, $c = f(O) = F(O)$, a constant, and α_i, p_i, and e_i satisfy the definitions of α, p, and e of the hypothesis. Let the equation of the plane P_i be $z = P_i(t, u)$, since no facet is perpendicular to the tu-plane, and let $f_1 = P_1(t, u) - c$ in S_1, $= 0$ in $D - S_1$. The line in which the plane, $z = P_1(t, u) - c$, cuts the tu-plane (since no facet need be parallel to the tu-plane*) passes through O, and may therefore be written $t\cos\alpha_1 + u\sin\alpha_1 = 0$. The equation of P_1, is therefore, $z = a_1\,(t\cos\alpha_1 + u\sin\alpha_1) + c$, and we may write:

$$\begin{aligned} f_1 &= a_1\,(t\cos\alpha_1 + u\sin\alpha_1) \text{ in } (e_1 = S_1), \\ &= 0 \text{ in } D - e_1. \end{aligned}$$

$$\text{(4)} \qquad \begin{aligned} z = c + f_1 &\text{ is } P_1 \text{ in } e_1, \\ &\text{is } \pi_1 \text{ in } \tau_1, \\ &\text{is } z = c \text{ in } D - S_1. \end{aligned}$$

Plane P_2 intersects P_1 in a line whose projection is $A'\,B'$. Let $f_2 = P_2(t, u) - P_1(t, u)$ in $(e_2 = S_1 - \tau_1)$, $= 0$ in $D - e_2$. Evidently $z = P_2 - P_1$ is the equation of a plane which intersects the tu-plane where $P_2 = P_1$, that is, in the line of intersection $A'\,B'$. Letting the equation of $A'\,B'$ be

* If a facet were, it could be replaced by two facets that would not be.

$t \cos \alpha_2 + u \sin \alpha_2 = p_2$, we may write:

$$f_2 = a_2 (t \cos \alpha_2 + u \sin \alpha_2 - p_2) \text{ in } (e_2 = S_1 - \tau_1),$$
$$= 0 \text{ in } D - e_2$$

$$z = c + f_1 + f_2 \text{ is } z = c + f_1, \text{ or } \pi_1, \text{ in } \tau_1, \tag{5}$$
$$\text{is } z = c + f_1 + P_2 - (c + f_1), \text{ or } P_2, \text{ in } e_2,$$
$$\text{is } \pi_2 \text{ in } \tau_2,$$
$$\text{is } z = c \text{ in } D - S_1.$$

Similarly $f_3 = P_3 - P_2 = a_3 (t \cos \alpha_3 + u \sin \alpha_3 - p_3)$ in $(e_3 = S_1 - \tau_1 - \tau_2)$,
$= 0$ in $D - e_3$,

etc.

The process may evidently be continued over S_1. In dealing next, in exactly the same manner, with S_2, we leave the function z unchanged in S_1, because the auxiliary functions f_i are zero outside the sector in question. The process may, therefore, be continued till we obtain the function $z = c + \sum_1^r f_i$, which is π_i in τ_i, and is the function F used above.

Now, since the following integrals exist,*

$$\int_D f\phi = \int_D (f - F)\,\phi + \int_D F\phi, \tag{6}$$

and since

$$\int_D F\phi = c \int_D \phi + \sum_{i=1}^{r} a_i \int_{e_i} (t \cos \alpha_i + u \sin \alpha_i - p_i)\,\phi,$$

in which by 2° the first term on the right-hand side approaches zero as n becomes infinite, and, r being fixed after ϵ is chosen, the second term approaches zero by 3°, it follows from 1°, (2), and (6) that

$$\left| \int_D f\phi \right| < \epsilon \int_D |\phi| + \left| \int_D F\phi \right| < \epsilon M + \epsilon \qquad (n, m, \cdots > n_{\epsilon, M}).$$

Necessity.—By referring to the proof of the necessary part of Theorem 4, it will be seen that 1° is necessary. If 2° is not satisfied, the theorem is contradicted by the function $f = 1$, and if 3° is not satisfied it is contradicted by the continuous function f defined as follows: $f = t \cos \alpha + u \sin \alpha - p$ in d, $= 0$ in $D - d$.

§ 5. *Explanation and proof of Theorem* 6.—Let $\{B_\lambda\}$ be a set of "monotone increasing point sets" in the field A. Such a set may conveniently be defined by means of an auxiliary function $\psi(t, u, \cdots)$. Let ψ be an arbitrary,

* See § 2, Lemma 1.

limited, L-integrable function, defined in A. Then, for each number λ, B_λ is the set of points where $\psi < \lambda$, and consequently $A - B_\lambda$ is the set where $\lambda \overline{\geq} \psi$. It is evident that all of $\{B_\lambda\}$ will be obtained if λ be confined to the interval * $(\min \psi, \max \psi + 1)$, for, if λ is not in this interval, B_λ either consists of no points or is A. We shall therefore restrict λ in this manner. Now f is to be regarded as "monotone increasing with respect to $\{B_\lambda\}$," provided it is defined and limited in A, and, ξ being an arbitrary number greater than $\min f$, the set of points E_ξ where $f < \xi$ is a B_λ. These provisions make f L-integrable. F_6, then, depends for its definition on the choice of ψ.

LEMMA: If $f(t, u, \cdots)$ *belongs to* F_6, *and if* $\phi(t, u, \cdots)$ *is absolutely* L*-integrable in* A,

$$\left|\int_A f\phi\right| \overline{\leq} 3 \max|f| \cdot \max_\lambda \left|\int_\lambda \phi\right|.$$

For, by a theorem of Lebesgue's,† using f, ϕ in place of his ϕ, f, respectively, we have

$$\left|\int_A f\phi\right| \overline{\leq} \max|f| \cdot \left(\max_\lambda \left|\int_{B_\lambda} \phi\right| + \max_\lambda \left|\int_{A-B_\lambda} \phi\right|\right) \overline{\leq} 3 \max|f| \cdot \max \left|\int_{B_\lambda} \phi\right|,$$

since evidently

$$\left|\int_{A-B_\lambda} \phi\right| \overline{\leq} \left|\int_A \phi\right| + \left|\int_{B_\lambda} \phi\right| \overline{\leq} 2 \max_\lambda \left|\int_{B_\lambda} \phi\right|.$$

The sufficiency of condition 1° follows directly from the lemma, and therefore we proceed at once to show that 2° and 3° are necessary.

If ϕ is not absolutely L-integrable, the function $f = 1$ belongs to F_6 and the integral over A of $f\phi$ does not exist as an absolutely convergent integral. If there exists a B_λ for which 3° is not satisfied, the function which equals -1 in this B_λ and zero in $A - B_\lambda$ contradicts the theorem. It only remains to show that the second part of 2° is necessary. To do this we shall assume a part of what has just been shown necessary, viz., that

(1) ϕ is absolutely L-integrable for each n, m, $\cdots$, and for each λ, H_λ exists ‡ so that

$$\left|\int_{B_\lambda} \phi\right| < H_\lambda \qquad (n, m, \cdots > n_H). \tag{2}$$

Let us suppose $U = \{B_\lambda\}$ to be the set of B_λ's for which the particular family

* See the third note to § 3.

† Annales de l'École Normale, loc. cit., p. 443. The notation $\max_\lambda u$ is used to signify the maximum of u regarded as a function of λ.

‡ We might assume here that H is arbitrarily small, i. e., 3°, but in the proof of Theorem 7 we need to have 2° proved necessary without this assumption.

F_6 in question exists, U being defined by some preassigned function ψ. Suppose the second part of 2° to be not satisfied. Then, for l and N arbitrary, there exist the numbers $n(l), m(l), \cdots > N$, and $\lambda(l)$, so that

$$\left|\int_{B_{\lambda(l)}} \phi[n(l), m(l), \cdots]\right| > l. \tag{3}$$

By (2) there exists a number M, which we may take > 1, and a corresponding n_M, so that

$$\left|\int_A \phi(n, m, \cdots)\right| < M \qquad (n, m, \cdots > n_M). \tag{4}$$

Now let $l = 2M > 2$, $N_1 = n_M$. By (3) there exist the numbers $n_1, m_1, \cdots > N_1 + 1$, and λ_1, so that

$$\left|\int_{B_{\lambda_1}} \phi(n_1, m_1, \cdots)\right| > 2M > 2. \tag{5}$$

Take, next,

$$l = \left|\int_{B_{\lambda_1}} \phi(n_1, m_1, \cdots)\right| + 1, \; N_2 = \max(n_1, m_1, \cdots) > N_1 + 1.$$

Then there exist the numbers $n_2, m_2, \cdots > N_2 + 2$, and λ_2, so that

$$\left|\int_{B_{\lambda_2}} \phi(n_2, m_2, \cdots)\right| > \left|\int_{B_{\lambda_1}} \phi(n_1, m_1, \cdots)\right| + 1. \tag{6}$$

Proceeding in this manner we have finally the following sequences, of which the first and third diverge to infinity:

$$\begin{array}{ccccccc} n_M < n_1, & m_1, & \cdots < n_2, & m_2, & \cdots < \cdots \\ & \lambda_1, & & \lambda_2, & & \cdots \end{array}$$

$$2 < 2M < \left|\int_{B_{\lambda_1}} \phi(n_1, m_1, \cdots)\right| < \left|\int_{B_{\lambda_2}} \phi(n_2, m_2, \cdots)\right| < \cdots \tag{7}$$

No λ_i is repeated an infinite number of times, for otherwise the absolute value of the integral over B_{λ_i} of $\phi(n, m, \cdots)$ diverges to infinity for a certain sequence of $(n, m, \cdots)$'s, and therefore the function f which equals -1 in B_{λ_i} and 0 in $A - B_{\lambda_i}$ contradicts the theorem. Consequently infinitely many of the λ_i's are distinct, and we may assume without loss of generality that they are all distinct.

Since the λ's are infinite in number they may be represented by a point-set in $(\min \psi, \max \psi + 1)$ which must have at least one limit point, L; and therefore there exists among the λ's an infinite set, $\lambda', \lambda'', \cdots$, having L as a limit point, such that, either

(a) $$\lambda' > \lambda'' > \cdots,$$

or

(b) $$\lambda' < \lambda'' < \cdots,$$

or both. By the definition of B_λ, then, either

(A) $$B_{\lambda'} \supseteq B_{\lambda''} \supseteq \cdots,$$

or

(B) $$B_{\lambda'} \subseteq B_{\lambda''} \subseteq \cdots.$$

Case B.—By (7) and (4), for an arbitrary l and an arbitrary N, there exists a $\lambda^{(i)}$ of (b) so that $n^{(i)}, m^{(i)}, \cdots > N$, and simultaneously

$$\left|\int_{B_{\lambda^{(i)}}} \phi(n^{(i)}, m^{(i)}, \cdots)\right| > 2M + l, \qquad \left|\int_A \phi(n^{(i)}, m^{(i)}, \cdots)\right| < M.$$

Applying these inequalities to the identity

$$\int_{A-B_{\lambda^{(i)}}} \phi(n^{(i)}, m^{(i)}, \cdots) = \int_A - \int_{B_{\lambda^{(i)}}},$$

we have

$$\left|\int_{A-B_{\lambda^{(i)}}} \phi(n^{(i)}, m^{(i)}, \cdots)\right| = \left|\int_A - \int_{B_{\lambda^{(i)}}}\right| > -M + 2M + l = M + l.$$

There exists in (b) a sequence $\lambda^{(i_1)}, \lambda^{(i_2)}, \cdots$ so that

$$\left|\int_{A-B_{\lambda^{(i_1)}}} \phi(n^{(i_1)}, m^{(i_1)}, \cdots)\right| < \left|\int_{A-B_{\lambda^{(i_2)}}} \phi(n^{(i_2)}, m^{(i_2)}, \cdots)\right| < \cdots,$$

$$A - B_{\lambda^{(i_1)}} \supseteq A - B_{\lambda^{(i_2)}} \supseteq \cdots,$$

and the former sequence diverges to infinity. By reference to (7) it is now clear that, without loss of generality, we may assume Case A; and, to save changing subscripts, let us suppose (a) and (A) to have been true of (7). Case B, continued, would mean that the g's defined later would be related to $A - B_\lambda$ as, in Case A, they are related to B_λ, except that the numbers would be positive instead of negative. Assuming Case A, then, we may add to (7):

(8) $$\lambda_1 > \lambda_2 > \cdots, \quad \text{and} \quad B_{\lambda_1} \supseteq B_{\lambda_2} \supseteq \cdots,$$

and here the λ's are all different and the first sequence has the limit L.

We are now ready to show the existence of a function g which contradicts the theorem. Setting $l_1 = 2M$, and so, rewriting

(5) $$\left|\int_{B_{\lambda_1}} \phi(n_1, m_1, \cdots)\right| > l_1 = 2M,$$

we note that, since, $n_1, m_1, \cdots$ are fixed, the following is, by (1) finite:

$$L_1 = \max_{\lambda} \left| \int_{B_\lambda} \phi(n_1, m_1, \cdots) \right|.$$

Let $g_1 = 0$ in $A - B_{\lambda_1}$, $= -1$ in B_{λ_1}. If the integral over A of $g_1 \phi(n, m, \cdots)$ does not approach zero as n becomes infinite, g_1 is the desired function, for B_{λ_1} belongs to U and therefore g_1 to F_6. Supposing, the worse case then, that this is not true, there exists a $\mu_1 > 0$ so that

$$\left| \int_A g_1 \phi(n, m, \cdots) \right| < 1 \qquad (n, m, \cdots > \mu_1). \tag{9}$$

Set

$$l_2 = 2^2 L_1. \tag{10}$$

By going far enough in (7) to a λ which, dropping intermediate λ's, we may rename λ_2, we have simultaneously

$$\left| \int_{B_{\lambda_2}} \phi(n_2, m_2, \cdots) \right| > 2^2 L_1 = l_2; \; n_2, m_2, \cdots > \mu_1. \tag{11}$$

Set

$$L_2 = \max_{\lambda} \left| \int_{B_\lambda} \phi(n_2, m_2, \cdots) \right|, \tag{12}$$

$$\begin{aligned} g_2 &= 0 \qquad \text{in } A - B_{\lambda_2}, \\ &= -\frac{2}{l_2} = \frac{-1}{2L_1} \text{ in } B_{\lambda_2}. \end{aligned} \tag{13}$$

By (11) and (8), respectively,

$$L_2 > L_1, \quad \text{and} \quad B_{\lambda_1} \leqq B_{\lambda_2}. \tag{14}$$

Consider the function $g_1 + g_2$.

In B_{λ_2}, $g_1 + g_2 = -1 - 2/l_2 < -1$.

In $A - B_{\lambda_2}$, $g_1 + g_2 =$ either -1, or 0.

In B_{λ_1}, $g_1 + g_2 =$ either $-2/l_2 - 1$, or -1, < 0.

In $A - B_{\lambda_1}$, $g_1 + g_2 = 0$.

For any number ξ which is greater than min $(g_1 + g_2)$, the set of points where $g_1 + g_2 < \xi$ is either B_{λ_2}, B_{λ_1}, or A, all of which belong to U. Hence $g_1 + g_2$ belongs to F_6. Therefore, if the integral over A of the product $(g_1 + g_2)\phi(n, m, \cdots)$ does not approach zero as n becomes infinite, $g_1 + g_2$ is the function desired.

Otherwise, there exists $\mu_2 > 0$ so that

$$\left| \int_A (g_1 + g_2) \phi(n, m, \cdots) \right| < 1 \qquad (n, m, \cdots > \mu_2). \tag{15}$$

Set

$$l_3 = 2^3 L_2, \text{ etc.} \tag{16}$$

Finally, if (the worst case) the number of steps is infinite, there exists a function g, limited in A,

$$g(t, u, \cdots) = g_1 + g_2 + \cdots, \tag{17}$$

where * $g_i = -2/l_i$ in B_{λ_i}, $= 0$ in $A - B_{\lambda_i}$; for (17) may be compared with the series,

$$-1 - \frac{2}{l_2} - \frac{2}{l_3} - \cdots = -1 - \frac{1}{2L_1} - \frac{1}{2^2 L_2} \cdots,$$

and this converges, since, by (14) and corresponding formulæ for the later indices, $L_1 < L_2 < \cdots$. Moreover, g belongs to F_6; for, by (8), we have for any i

$$\text{in } B_{\lambda_i},\ g \overline{\overline{<}}\ g_1 + \cdots + g_i = -1 - \frac{2}{l_2} - \cdots - \frac{2}{l_i} < -1 - \frac{2}{l_2} - \cdots - \frac{2}{l_{i-1}},$$

$$\text{in } A - B_{\lambda_i},\ g = g_1 + \cdots + g_i \overline{\overline{>}} -1 - \frac{2}{l_2} - \cdots - \frac{2}{l_{i-1}},$$

and, for any ξ such that $\min g < \xi < \max g + 1$, B_ξ is some B_{λ_i}, which belongs to U. It only remains to show, then, that

$$\int_A g\phi(n, m, \cdots) \tag{18}$$

does not converge to zero as n becomes infinite.

For each subscript $j > 2$,

$$\int_A g\phi(n_j, m_j, \cdots) = \sum_{i=1}^{j-1} \int_A g_i \phi(n_j, m_j, \cdots) + \int_A g_j \phi(n_j, m_j, \cdots) + \sum_{i=j+1}^{\infty} \int_A g_i \phi(n_j, m_j, \cdots) = \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

By (15) etc., $|\mathrm{I}| < 1$.

By (13) etc., and by (11) etc.,

$$|\mathrm{II}| = \left| \int_{B_{\lambda_j}} \frac{1}{2^{j-1} L_{j-1}} \phi(n_j, m_j, \cdots) \right| > \frac{2^j L_{j-1}}{2^{j-1} L_{j-1}} = 2.$$

By successive applications of (13) etc., (12) etc., and (14) etc., we have

$$|\mathrm{III}| = \left| \frac{1}{2^j L_j} \int_{B_{\lambda_{j+1}}} \phi(n_j, m_j, \cdots) + \frac{1}{2^{j+1} L_{j+1}} \int_{B_{\lambda_{j+2}}} \phi(n_j, m_j, \cdots) + \cdots \right|$$

$$\overline{\overline{<}} \left(\frac{1}{2^j L_j} + \frac{1}{2^{j+1} L_{j+1}} + \cdots \right) L_j < \frac{1}{2^j} + \frac{1}{2^{j+1}} + \cdots < \tfrac{1}{2} \quad (j > 2).$$

* Except that $g_1 = -2M/l_1$.

Hence, if $j > 2$,

$$|\mathrm{I} + \mathrm{II} + \mathrm{III}| > 2 - 1 - \tfrac{1}{2} = \tfrac{1}{2},$$

and (18) is proved.

§6. Let us now consider the integral

$$\int_E f(t, u)\, \theta(t, u; x, y; n, m)\, dE$$

and its convergence to $f(x, y)$ at the arbitrary, fixed point (x, y) of the limited, measurable field E, in space of two * dimensions.

THEOREM 7: *Let (a) f be continuous at (x, y), (b) f belong to one of the families F of the previous theorems, (c) E satisfy the conditions imposed on the field of integration in that one of the preceding theorems which relates to the F in question, (d) V_δ be a circle of radius $\delta > 0$, whose center is (x, y), and suppose also that E is dense and metric in some V_δ.*

Conditions that this integral may approach $f(x, y)$, as n becomes infinite, for all f's belonging to one of the families F, may be stated in the several cases as follows:

If F is F_1, F_1', F_2, F_3, or F_4, it is necessary and sufficient that there exist a δ_0 so that, for every positive $\delta < \delta_0$, (1°) the conditions of the previous theorem relating to the F in question are satisfied for $E - V_\delta$, that (2°) there exist an M so that the integral over V_{δ_0} of the absolute value of θ is less than M, if $n, m > n_M$, and that (3°) the integral over E of θ approach unity as n becomes infinite.

If F is F_6 or F_6', it is sufficient that the sufficient conditions involved in 1° be satisfied with respect to B_λ', where B_λ' is the part of B_λ in $E - V_\delta$, that 3° be satisfied, and that (2a°) there exist an M so that the absolute value of the integral over B_λ'' of θ is less than M, if $n, m > n_M$, for all δ's $\leqq \delta_0$ and all λ's uniformly, where B_λ'' is the part of B_λ in V_δ.

If F is F_5, it is necessary and sufficient that the necessary and sufficient conditions, respectively, of Theorem 5 be satisfied relative to E, 3° replacing 2° of Theorem 5, and the region d being defined by using (x, y) as a new origin; this imposes a certain restriction on E (cf. the second sentence in § 4).

The proof of this theorem follows from the preceding theorems in much the same manner as the proof of the corresponding theorem † in one dimension follows from their analogues.

APPLICATIONS TO SERIES OF NORMAL FUNCTIONS.

§7. THEOREM 8: *Let $w_i(t, u, \cdots)$ be a complete set of normal ‡ functions in*

* The generalization to more than two dimensions in certain cases is evident from the preceding theorems. See also the first note to § 1.

† Annales de la Faculté de Toulouse, loc. cit., p. 70. Note also the remark concerning the integral on page 28.

‡ I. e., orthogonal functions of which the integrals of the squares over the field in question are equal to unity. Consult, e. g., RIESZ, next footnote.

the limited, measurable field A. In order that the coefficients of the formal development of $f(t, u, \cdots)$ in terms of the w's may exist for sufficiently large values of i and approach zero as i becomes infinite, for all functions f of each of the families F_1, F_2, F_3, F_4, and F_5, previously considered, it is necessary and sufficient that in each case there exist an M so that when $i > i_M$ the inequality mentioned below for the family in question is satisfied:

If F is F_1, (1°) $|w_i|$ shall be less than M.

If F is F_2, (2°) the integral over A of w_i^2 shall be less than M.

If F is F_3, (3°) the integral over A of $|w_i|$ shall be less than M.

If F is F_4, or F_5, (4°) the integral over C of $|w_i|$ shall be less than M.

COROLLARY: *Similar statements may be made for functions of one variable, and in that case we have the additional statement that if f is a function of limited variation, it is necessary and sufficient that M exist so that, if $i > i_M$, the absolute value of the integral from a to λ of w_i is less than M for all λ's in (a, b), (a, b) being the interval in which the functions are defined.*

It will be shown that if F is F_1, 1° is necessary and sufficient. If F is some other family, the proof is analogous.

Sufficiency.—F. RIESZ * has shown that if $\{w_i\}$ is as in our hypothesis, and if f^2 (and therefore, by SCHWARZ's inequality,† f) is absolutely L-integrable, the series of the squares of the coefficients,

$$\sum_{i=1}^{\infty}\left[\int_A f w_i\right]^2$$

is convergent. Hence, under these conditions, each coefficient, (1) $\int_A f w_i$, approaches zero as i becomes infinite, and this is true in particular when $f = 1$ in a and zero elsewhere, a being any measurable component of A. That is, provided only that $\{w_i\}$ be as above, (2) $\int_a w_i$ approaches zero as i becomes infinite. This is condition 2° of Theorem 1, and in our hypothesis 1° is condition 1° of that theorem, w_i being a special case of $\phi(n, m, \cdots)$. Hence (1) follows from 1° and (2).

Necessity.—If w_i does not satisfy 1°, then by Theorem 1 there exists an f belonging to F_1 for which the integral over A of $f w_i$ does not approach zero as i becomes infinite. Therefore 1° is necessary.

§ 8. DOUBLE FOURIER SERIES: Let $f(t, u)$ have the double period 2π in the tu-plane, and be absolutely L-integrable in the square S, where S is $(x - \pi \leqq t \leqq x + \pi,\ y - \pi \leqq u \leqq y + \pi)$, and (x, y) is an arbitrary,

* Comptes Rendus, vol. 144 (1907), p. 616, p. 734. RIESZ does not state that f^2 is supposed L-integrable, but he implies it in his proof.

† Annales de l'École Normale, loc. cit., p. 442.

fixed point. We may formally develop $f(x, y)$ in a double series of the form,

$$\sum_{i,j=1}^{\infty} [(a_{ij} \sin ix + b_{ij} \cos ix) \sin jy + (c_{ij} \sin ix + d_{ij} \cos ix) \cos jy].$$

As is well known, it may be shown in a manner analogous to the method used in simple series that the sum of all those terms whose subscripts are not greater than n, m, respectively, may be put in the form:

$$S(n, m; x, y) = \frac{1}{\pi^2} \int_S f(t, u) \left[\frac{\sin \frac{1}{2}(2n+1)(t-x)}{2 \sin \frac{1}{2}(t-x)} \frac{\sin \frac{1}{2}(2m+1)(u-y)}{2 \sin \frac{1}{2}(u-y)} \right].$$

We now define the sum of the double series to mean the double * limit of $S(n, m)$, if this limit exists, as n, m become infinite. We will show that, under certain circumstances, this limit does exist and is equal to $f(x, y)$. In this integral we make the substitutions:

$$t - x = 2\alpha', \; u - y = 2\beta'; \; 2n + 1 = \nu, \; 2m + 1 = \mu.$$

S becomes the square $(-\pi/2 \leqq \alpha', \beta' \leqq \pi/2)$. Then, making the transformation,

$\alpha = \alpha', \quad \beta = \beta', \quad$ where $0 \leqq \alpha', \beta' \leqq \pi/2,$

$\alpha = -\alpha', \quad \beta = \beta', \quad$ where $-\pi/2 \leqq \alpha' \leqq 0$ and $0 \leqq \beta' \leqq \pi/2,$

$\alpha = -\alpha', \quad \beta = -\beta', \quad$ where $-\pi/2 \leqq \alpha', \beta' \leqq 0,$

$\alpha = \alpha', \quad \beta = -\beta', \quad$ where $0 \leqq \alpha' \leqq \pi/2$ and $-\pi/2 \leqq \beta' \leqq 0,$

and proceeding in a manner analogous to the well-known † theory for simple Fourier series, we may learn that the validity of the expansion depends on the convergence to zero, as ν becomes infinite, of the integral

$$D(\nu, \mu) = \int_{S_1} \frac{\sin \nu\alpha \sin \mu\beta}{\sin \alpha \sin \beta} \phi(\alpha, \beta), \text{ where}$$

$$(2) \quad \phi(\alpha, \beta) = f(x - 2\alpha, y - 2\beta) + f(x + 2\alpha, y - 2\beta) + f(x - 2\alpha, y + 2\beta) + f(x + 2\alpha, y + 2\beta) - 4f(x, y),$$

and S_1 denotes the square, $(0 \leqq \alpha, \beta \leqq \pi/2)$.

The substitutions made above are valid if the various integrals may be expressed as iterated integrals,‡ a condition which is fulfilled since the double

* Cf. the first footnote to this paper.

† E. g., PIERPONT, loc. cit., vol. 2, pp. 421–3.

‡ Annales de l'École Normale, loc. cit., pp. 447–50.

integrals * exist, f being absolutely L-integrable † and $\sin \nu\alpha / \sin \alpha$, $\sin \mu\beta / \sin \beta$ being limited for each ν, μ.

THEOREM 9: *If $f(t, u)$ is absolutely L-integrable in S, its double Fourier series is valid at the fixed point (x, y), provided, in (2), $\phi(\alpha, \beta) / \alpha\beta$ is absolutely L-integrable in S_1.*

Proof.—Writing $D(\nu, \mu)$ in the form,

$$\int_{S_1} \frac{\phi(\alpha, \beta)}{\alpha\beta} \frac{\alpha \sin \nu\alpha \cdot \beta \sin \mu\beta}{\sin \alpha \sin \beta},$$

we learn from Theorem 1 that it is sufficient to show that

$$\left| \frac{\alpha \sin \nu\alpha \cdot \beta \sin \mu\beta}{\sin \alpha \sin \beta} \right| < \text{some } M, \text{ for all } \nu, \mu, \tag{3}$$

and

$$\int_q \frac{\alpha \sin \nu\alpha \cdot \beta \sin \mu\beta}{\sin \alpha \sin \beta} \tag{4}$$

approaches zero for each q in S_1, as ν, μ become infinite. Since $1 \leqq \alpha / \sin \alpha \leqq \pi/2$, α in $(0, \pi/2)$, and $|\sin \nu\alpha| \leqq 1$, (3) is evident. We may write (4) in the forms,

$$\int_a^b \frac{\alpha \sin \nu\alpha}{\sin \alpha} d\alpha \int_a^b \frac{\beta \sin \mu\beta}{\sin \beta} d\beta = \left[\int_a^b \frac{\alpha \sin \nu\alpha}{\sin \alpha} d\alpha \right] \left[\int_a^b \frac{\beta \sin \mu\beta}{\sin \beta} d\beta \right],$$

since the double integral exists, and the limits, a, b, are constant. Now $\alpha / \sin \alpha$ is monotone increasing in $(0, \pi/2)$, and $\sin \nu\alpha$ is L-integrable. Hence the lemma of § 5 applies, and

$$\left| \int_a^b \frac{\alpha \sin \nu\alpha}{\sin \alpha} d\alpha \right| \leqq 3 \max \left| \frac{\alpha}{\sin \alpha} \right| \cdot \max_\lambda \left| \int_a^\lambda \sin \nu\alpha \, d\alpha \right|$$

$$\leqq 3 \cdot \frac{\pi}{2} \cdot \frac{1}{\nu} \max_\lambda |\cos \nu a - \cos \nu\lambda| \leqq \frac{3\pi}{\nu}.$$

$$\left| \int_a^b \frac{\alpha \sin \nu\alpha}{\sin \alpha} d\alpha \int_a^b \frac{\beta \sin \mu\beta}{\sin \beta} d\beta \right| \leqq \frac{9\pi^2}{\nu\mu},$$

and (4) is proved.

COROLLARY: *If f is absolutely L-integrable in S, the validity of the expansion at (x, y) depends only on the behavior of the function in the vicinity of the lines through this point parallel to the coördinate axes.*

For it is only in this vicinity that additional restrictions on f are needed in order to make $\phi / \alpha\beta$ integrable. Cf. Lemma 1, § 2.

* HOBSON, Proceedings of the London Mathematical Society, ser. 2, vol. 8 (1910), p. 30.

† See Lemma 1, § 2.

THEOREM 10: *If f is absolutely L-integrable in S, the coefficients of its formal series at (x, y) approach zero as a double limit as n, m become infinite.*

For $\sin nt$ and $\cos nt$ are numerically less than or equal to unity for all values of n and t. Cf. Theorem 8, 1°.

REMARK: These theorems include corresponding theorems in simple Fourier series (already known) as special cases, and may be stated more generally so as to apply to multiple Fourier series of functions having any desired periods in the i variables, $t, u, \cdots$.

DEFINITE INTEGRALS.

§ 9. Other applications of the fundamental theorems may be found in the evaluation of absolutely convergent, multiple L-integrals of the form,

$$\int_A f(t, u, \cdots) g(t, u, \cdots) dA,$$

by means of the device of expanding one of these functions, say g, in a multiple series, multiplying through by f, and integrating termwise. If the partial sum of such a series be denoted by $s_{n,m,\ldots}$, it is clear that this process is equivalent to the requirements that the integral over A of the product, $f(g - s_{n,m,\ldots})$, exist for sufficiently large values of $n, m, \cdots$, and that its multiple limit be zero. In Theorems 1–6, therefore, replacing $\phi(n, m, \cdots)$ by $g - s_{n,m,\ldots}$, the reader will find stated the necessary and sufficient conditions that the process be allowable.

It will be noticed that condition 2° is satisfied if the series for g converges, except perhaps for a null set, and is integrable termwise in each set a. VITALI* has shown that a simple series of limited functions of one variable has this property if it converges to an integrable function, and if condition $1a°$ of Theorem 3 (cf. Corollary 1, Theorem 3, and footnote) is satisfied. His reasoning is sufficient, however (by the aid of a theorem of Lebesgue†), to establish the same result for simple series of absolutely L-integrable functions of i variables defined in A, whether these functions are limited or not. Therefore, at least if simple series be used, the conditions that the above process of evaluation be valid, whenever the g series converges to g (except perhaps at a null set), and all the integrals involved exist, may be stated more simply in certain cases, as follows:

If f is in F_1, it is sufficient that $|g - s_n|$ be less than M, except perhaps at a null set, when $n > n_M$.

*Rendiconti di Palermo, vol. 23 (1907), p. 139. VITALI uses $1a°$ with reference to s_n instead of $g - s_n$, but it is readily shown that there is no difference, since g is integrable.

†Annales de l'École Normale, loc. cit., p. 375, § 15. A fundamental theorem in the theory of measurable sets is also needed. Cf. PIERPONT, loc. cit., vol. 2, p. 358.

If f is in F_2, it is sufficient that the integral over A of $(g - s_n)^2$ be less than M, when $n > n_M$.

If f is in F_3, it is sufficient that for each $\gamma > 0$ there exist two numbers, $\lambda_\gamma > 0$, and n_γ, so that for every measurable set b in A whose measure is less than λ_γ, the integral over b of the absolute value of $(g - s_n)$ is less than γ when $n > n_\gamma$.

For it has previously been shown that if 1° of either Theorem 1 or Theorem 2 is satisfied, so is $1a$° of Theorem 3.

These conditions are also *necessary* in the same sense as the conditions of Theorem 1 are necessary; that is, if, for example, the first condition is not satisfied, then there exists an f belonging to F_1 for which the process is not valid.

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DECOMPOSITION OF AN N-SPACE BY A POLYHEDRON*

BY

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1. The following pages contain a proof of the theorem that an $(n-1)$-dimensional polyhedron decomposes an n-dimensional space into two regions. The proof has an essentially combinatorial character and involves few geometrical ideas beyond the principle than an n-dimensional convex region is decomposed into two n-dimensional convex regions by an $(n-1)$-space which contains one of its points and lies in the same n-space.

The theorem has been proved for the case $n = 2$ by H. HAHN † and for $n = 2, 3$ by N. J. LENNES.‡ The n-dimensional case, for polyhedra without singularities (cf. § 5 below), can be taken as a corollary under the theorem on the decomposition of an n-space by an $(n-1)$-dimensional manifold as proved by L. E. J. BROUWER.§ Our proof however allows a more general class both of spaces ‖ and of polyhedra within a given space.

2. The n-dimensional space, S_n, in which we are working may be defined as a "number space," or it may be taken as defined by a set of axioms. We shall adopt the latter point of view as being more convenient and general. For analytic purposes it is only necessary to verify the obvious fact that the axioms are satisfied by a number space.

We shall take as axioms, I–VIII of my thesis ¶ together with the further assumption (necessary for n-dimensional geometry) that there exists a set of $n+1$ points not all in the same $(n-1)$-dimensional space. These axioms do not include an Archimedean axiom or any assumption about continuity.

3. We take for granted the definition and properties of the n-dimensional

* Read before the Society, September 11, 1912.

† Monatshefte für Mathematik und Physik, vol. 19 (1908), pp. 289–303. In this article HAHN points out that the proof in my thesis, these Transactions, vol. 5 (1904), p. 365 (Th. 28), is incorrect. The same mistake occurs in the proof of Theorem 9, p. 93 of my article, *Theory of Plane Curves in Non-Metrical Analysis Situs*, these Transactions, vol. 6 (1905), p. 93. Theorem 9 is a corollary of the theorem that a simple polygon decomposes the plane into two regions, so that, a correct proof of the polygon theorem having been supplied, this mistake does not affect the rest of the paper.

‡ American Journal of Mathematics, vol. 33 (1911), pp. 37–62.

§ Mathematische Annalen, vol. 71 (1911), p. 314.

‖ We do not assume continuity.

¶ These Transactions, vol. 5 (1904), pp. 343–384.

simplex, the generalization of the triangle and tetrahedron. This is a set of $n+1$ points (not all in the same S_{n-1}) together with the $(n+1)\,n/2$ segments which they determine by pairs, the $(n+1)\,n\,(n-1)/2.3$ triangular regions which they determine by trios, $\cdots$, the $(n+1)n\cdots 2/(n)!$ interiors of $(n-1)$-dimensional simplexes which they determine by sets of n. The simplex separates the points of S_n into two sets, its interior and exterior, such that any broken line joining a point of one set to a point of the other meets the simplex, such that any segment joining two points of the interior consists entirely of points of the interior, and such that the exterior contains an S_{n-1} consisting entirely of points of the exterior.

The details of this discussion are analogous to those given for the triangle and tetrahedron in my thesis. A like remark applies to the contents of §§ 4, 5, 6 below.

4. An *n-dimensional region* is defined as a set of points all in the same S_n such that any two points of the set are joined by a broken line consisting entirely of points of the set and such that any point of the set is interior to an n-dimensional simplex consisting entirely of points of the set. A region is *convex* if the segment joining any two of its points consists of points of the region. The *boundary* of an n-dimensional region is a set of points $[B]$ such that (1) any broken line lying entirely in the S_n which contains the region and joining a point of the region to a point not of the region must contain a point B, and (2) any subset of $[B]$ does not have the property (1). According to these definitions a region cannot contain any point of its boundary.

5. A one-dimensional region is either a segment,* a ray (half-line) not including its end point, or a complete line. A two-dimensional region whose boundary consists of a finite number of points and one-dimensional regions is called a two-dimensional *polygonal region*.†

An n-dimensional region whose boundary consists of a finite number of points, one-dimensional regions, two-dimensional polygonal regions, $\cdots$, $(n-1)$-dimensional polyhedral regions is called an n-dimensional *polyhedral region*.‡

An *n-dimensional polyhedron* is the set of points in the interiors and boundaries of a finite set of n-dimensional polyhedral regions $F_1, F_2, \cdots, F_k$ no two of which have a point in common but such that (1) every $(n-1)$-dimensional polyhedral region on the boundary of one n-dimensional region F is on the boundary of an even number of n-dimensional polyhedral regions

* A segment does not include its end-points; an interval does.

† For example, a complete plane or the two sides of a line in a plane are polygonal regions according to this definition. Throughout this paper there is no restriction as to figures lying in "a finite part of space."

‡ It is an immediate corollary of this definition that the points, if any, common to two polyhedral regions constitute one or more polyhedral regions.

F and (2) there is no subset of the points constituting the polyhedron which can be arranged in n-dimensional polyhedral regions so as to have the property (1).* The regions $F_1, \cdots, F_k$ are called the *faces* of the polyhedron.

According to this definition a polyhedron can have rather complicated singularities in all cases where $n > 2$. An n-dimensional polyhedron without singularities is obtained by requiring, in addition to conditions (1) and (2), that a complete neighborhood of any point on it be capable of being set in one-to-one correspondence, preserving order relations, with the interior of an $(n-1)$-dimensional simplex. In this paper, however, we are concerned with the general polyhedron, allowing singularities.

6. Let us now consider an $(n-1)$-dimensional polyhedron in an n-space S_n. Let us denote the faces of the polyhedron by $F_1, F_2, \cdots, F_k$. Let us denote the $(n-1)$-spaces which contain them by $S^1_{n-1}, S^2_{n-1}, \cdots, S^j_{n-1}$. There are not necessarily as many of these as of the F's since two F's may be in the same S_{n-1}.

By an argument which is a mere generalization of that in the proof of Theorem 26 of my thesis, the j S_{n-1}'s decompose S_n into a certain number, N, of mutually exclusive convex polyhedral regions

$$C_1, C_2, \cdots, C_N.$$

Each of the S_{n-1}'s is met by the other S_{n-1}'s in a number of S_{n-2}'s which decompose it into a number of convex $(n-1)$-dimensional polyhedral regions. Let us denote these by

$$a_1, a_2, \cdots, a_M,$$

M being the number of a's on all the S_{n-1}'s. The boundaries of the C's may be denoted by

$$b_1, b_2, \cdots, b_N$$

respectively. The b's are polyhedra. Any one of the a's is in two and only two b's. Moreover a single b has at most one region a in common with a given S_{n-1}.

The foregoing statements hold good for any finite number of S_{n-1}'s in the same S_n, except that, in general, each b is a set of polyhedra instead of a single polyhedron.

7. *If π is any polyhedron composed entirely of points of $S^1_{n-1}, S^2_{n-1}, \cdots, S^j_{n-1}$, and a_k is any one of the convex $(n-1)$-dimensional regions a, then if π contains one point of a_k it contains all points of a_k.*

In proving this we may assume that a_k is on S^j_{n-1}. No point of a_k is on $S^1_{n-1}, \cdots, S^{j-1}_{n-1}$. Hence if π contains a point of a_k, this point (which we shall

* This definition of a polyhedron is a generalization of that used by LENNES (*loc. cit.*) in the one- and two-dimensional cases.

call P) may be taken as interior to an $(n-1)$-dimensional polygonal region F of π which is contained entirely in S^j_{n-1}. If π does not contain all points of a_k there must be an $(n-1)$-dimensional simplex whose interior consists of points of a_k not in π. Let Q be one of these points such that the linear segment PQ does not meet any one of the finite number of polyhedral regions of dimensionality $(n-3)$ or less which bound the $(n-1)$-dimensional polyhedral regions of π. The linear interval PQ meets the $(n-1)$-dimensional regions of π and their boundaries in a finite number of points and segments. Let N be the last one of the finite set composed of these points and of the end points of these segments in the sense from P to Q. The point N must be on the boundary of an $(n-1)$-dimensional region F' of π; and since Q was so chosen that N must be interior to an $(n-2)$-dimensional region of the boundary of F', any other $(n-1)$-dimensional region of π in S^j_{n-1} which had N on its boundary would contain points of the segment NQ, contrary to the definition of N. Since N is in a it is not on any of $S^1_{n-1}, \cdots, S^{j-1}_{n-1}$, and hence cannot be interior to any $(n-1)$-dimensional region of π which is not in S^j_{n-1}. Hence N is on the boundary of only one $(n-1)$-dimensional region of π, contrary to the definition of a polyhedron. Hence π contains all points of a_k.

Every point of π which is on S^j_{n-1} is either interior to, or on the boundary of one of the convex regions $a_{k_1}, a_{k_2}, \cdots$ into which S^j_{n-1} is decomposed by $S^1_{n-1}, \cdots, S^{j-1}_{n-1}$. Hence what we have just proved shows that the regions F of π on S^j_{n-1}, together with their boundaries, constitute the interiors and boundaries of a finite number of a_k's. We shall denote these a_k's by $a_{k_1}, a_{k_2}, \cdots, a_{k_\nu}$. Any $(n-2)$-dimensional convex portion of the boundary of one of $a_{k_1}, a_{k_2}, \cdots, a_{k_\nu}$ is either on one and only one other of $a_{k_1}, a_{k_2}, \cdots, a_{k_\nu}$, or else it is also a portion of the boundary of one of the F's in S^j_{n-1}. Hence if the F's in S^j_{n-1} are replaced by $a_{k_1}, a_{k_2}, \cdots, a_{k_\nu}$, the latter together with the F's in $S^{j-1}_{n-1}, S^{j-2}_{n-1}, \cdots, S^1_{n-1}$ form a set of $(n-1)$-dimensional polyhedral regions which satisfy the definition of a polyhedron.

In like manner, the $(n-1)$-dimensional regions of π in $S^{j-1}_{n-1}, \cdots, S^1_{n-1}$ can be replaced by convex regions. Hence *any polyhedron π composed entirely of points of $S^1_{n-1}, S^2_{n-1}, \cdots, S^j_{n-1}$ can be regarded as a polyhedron whose faces are a subset of the convex $(n-1)$-dimensional regions, $a_1, a_2, \cdots, a_M$.*

8. Given any set whatever of the convex $(n-1)$-dimensional regions, $a_1, a_2, \cdots, a_M$, let us introduce the symbol

$$(x_1, x_2, \cdots, x_M),$$

in which x_i $(i = 1, 2, \cdots, M)$ is 1 if a_i is in the set and 0 if x_i is not in the set. This symbol may be used to denote the totality of points in the set of a's and on their boundaries. According to § 7 any $(n-1)$-dimensional polyhedron composed entirely of points of $S^1_{n-1}, \cdots, S^j_{n-1}$, may be denoted

by a symbol of this form. By the definition of a polyhedron the sum of the x's corresponding to the a's having a given $(n-2)$-dimensional region on their boundaries is even or zero.

Let us agree to combine the x's by addition and multiplication, reducing modulo two. The symbol,

$$x + y = (x_1 + y_1,\ x_2 + y_2,\ \cdots,\ x_M + y_M),$$

(read, the *sum* of x and y) will then stand for the set of points within and on the boundaries of the a's obtained by combining the a's (represented by $x = (x_1, x_2, \cdots, x_M)$ with those represented by $y = (y_1, y_2, \cdots, y_M)$ and suppressing those which appear in both sets.

According to this definition, the sum of two polyhedra is always one or more polyhedra, as follows directly from the definition of a polyhedron. Moreover we always have

$$x + x = (0, 0, \cdots, 0).$$

The essential feature of our argument will be to show that the symbol for any polyhedron can be expressed as a sum of the symbols for a certain set of the b's. We shall use $b_1, b_2, \cdots, b_N$ as abbreviations for the expressions of the form $(x_1, x_2, \cdots, x_M)$ which represent the boundaries $b_1, b_2, \cdots, b_N$ respectively.

9. If S_{n-1}^j be removed, the remaining S_{n-1}'s, $S_{n-1}^1, \cdots, S_{n-1}^{j-1}$ decompose S_n into a number of convex regions some of which are identical with some of the C's and some of which are such that they are decomposed into pairs of C's by S_{n-1}^j. The boundary of any one of the latter regions is expressible, according to the above convention, in the form

$$b_\gamma + b_\delta,$$

where C_γ and C_δ are the two convex regions into which the given convex region is divided by S_{n-1}^j.

By a repetition of this reasoning it is established that if any subset of the S_{n-1}'s be removed, the boundaries of the n-dimensional convex regions into which S_n is decomposed by the remaining S_{n-1}'s are linearly expressible in terms of the b's.

10. Since every $(n-1)$-dimensional convex region a on each b is on one and only one other b,

$$b_1 + b_2 + \cdots + b_N = (0, 0, \cdots, 0), \tag{1}$$

where each b enters once and only once into the summation.

Moreover there exists no relation of the form

$$b_\alpha + b_\beta + \cdots + b_\nu = (0, 0, \cdots, 0) \tag{2}$$

which does not include all the b's. For consider the corresponding C's and let P be a point in one of these C's and Q a point in one of the C's not in this set. Let P and Q be so chosen that the line PQ does not meet any of the finite number of convex regions of dimensionality $(n-2)$ or less which form parts of the b's. Then the segment PQ meets the $b_\alpha, b_\beta, \cdots, b_\nu$ in a finite number (>0) of points. Let K be the last of these in the sense from P to Q. The segment KQ is not in any of $C_\alpha, C_\beta, \cdots, C_\nu$, and hence the $(n-1)$-dimensional region of the boundary of the one of $b_\alpha, b_\beta, \cdots, b_\nu$ which contains K is on one and only one b. This contradicts Equation (2).

11. *Any $(n-1)$-dimensional polyhedron π composed entirely of points of $S^1_{n-1}, \cdots, S^j_{n-1}$ is expressible in the form*

$$\pi = b_{\nu_1} + b_{\nu_2} + \cdots + b_{\nu_r}. \tag{3}$$

This is proved by mathematical induction. In the first place, if there is only one S_{n-1} it constitutes the only $(n-1)$-dimensional polyhedron composed entirely of points of itself and is itself the boundary of a convex region. Hence by § 9 it is linearly expressible in terms of the b's.

Now supposing the theorem true for $S^1_{n-1}, \cdots, S^i_{n-1}$, any $(n-1)$-dimensional polyhedron composed of points from these S_{n-1}'s is expressible linearly in terms of the boundaries of the convex regions into which S_n is decomposed by $S^1_{n-1}, \cdots, S^i_{n-1}$, and hence by § 9 in terms of $b_1, b_2, \cdots, b_N$. Any $(n-1)$-dimensional polyhedron, π, composed of points of $S^1_{n-1}, \cdots, S^i_{n-1}, S^{i+1}_{n-1}$ is either of the sort considered in the last sentence or contains one or more of the convex $(n-1)$-dimensional polyhedral regions, R, into which S^{i+1}_{n-1} is decomposed by $S^1_{n-1}, \cdots, S^i_{n-1}$.

By § 7 these R's may be regarded as the faces of π in S^{i+1}_{n+1}. Any R is in the boundary of just two of the n-dimensional convex regions determined by $S^1_{n-1}, S^2_{n-1}, \cdots, S^{i+1}_{n-1}$ (§ 6). By § 9 the boundary of either of these regions is expressible in the form $b_{\lambda_1} + b_{\lambda_2} + \cdots + b_{\lambda_\rho}$. By § 6 the polyhedron or set of polyhedra represented by $b_{\lambda_1} + b_{\lambda_2} + \cdots + b_{\lambda_\rho}$ has no other face in common with S^{i+1}_{n+1}. Hence $\pi + b_{\lambda_1} + b_{\lambda_2} + \cdots + b_{\lambda_\rho}$ is a polyhedron or set of polyhedra which has in common with S^{i+1}_{n-1} one $(n-1)$-dimensional convex region fewer than π has. By repeating this process a finite number of times we finally arrive at a polyhedron or set of polyhedra,

$$\pi + b_1 + b_2 + \cdots + b_r,$$

which has no $(n-1)$-dimensional regions in common with S^{i+1}_{n-1}. Such polyhedra have already been seen to be linearly expressible in terms of the b's. But since the relation

$$\pi + b_1 + \cdots + b_r = b_{r+1} + \cdots + b_{r+s}$$

is equivalent (§ 8) to

$$\pi = b_1 + \cdots + b_r + b_{r+1} + \cdots + b_{r+s},$$

we have the result that every $(n-1)$-dimensional polyhedron composed entirely of points of $S^1_{n-1}, \cdots, S^{i+1}_{n-1}$ is expressible in this form.

12. By adding Equation (1) in § 8 to Equation (3) in § 10 we obtain a second equation for π,

$$\pi = b_{\mu_1} + b_{\mu_2} + \cdots + b_{\mu_t}. \tag{4}$$

Every b appears once and only once in the two equations, (3) and (4). There is no other linear expression for π in terms of the b's. For if there were, on adding it to (3) the sum would give a relation of the form (2) which has been shown not to exist.

13. Consider now the set of n-dimensional convex regions $C_{\nu_1}, C_{\nu_2}, \cdots, C_{\nu_r}$ corresponding to the b's in (3), and let $[P]$ be the set of points interior to and on the boundaries of these C's but not on π. I say that any two points of $[P]$ can be joined by a broken line consisting of points P.

For let P_0 be an arbitrary point interior to C_{ν_1}. P_0 can be joined by an interval consisting entirely of points P to all points of the interior and boundary of C_{ν_1} which are not on π. Unless π includes all the $(n-1)$-dimensional convex regions, a, of the boundary of C_{ν_1} (in which case π would be identical with b_{ν_1}, by the definition of the polyhedron, and $[P]$ would be identical with C_{ν_1}) the boundary of b_{ν_1} has an a in common with the boundary of another C, say with b_{ν_2}. Let a_{ν_1} be the $(n-1)$-dimensional region common to b_{ν_1} and b_{ν_2} and let P_1 be any point of a_{ν_1}.

The broken line $P_0 P_1 Q$ will join P_0 to any point, Q, of the interior or boundary of C_{ν_2} which is not on π. This broken line consists entirely of points of $[P]$. Unless $\pi = b_{\nu_1} + b_{\nu_2}$, there is some $(n-1)$-dimensional convex region a_{ν_2} common to $b_{\nu_1} + b_{\nu_2}$ and one (say b_{ν_3}) of $b_{\nu_3}, \cdots, b_{\nu_r}$; for the equation $\pi = b_{\nu_1} + \cdots + b_{\nu_r}$ means that every $(n-1)$-dimensional region a of each b is either a region of π or of another b_ν. If P_2 is any point of a_{ν_2}, the broken line joining P_0 to P_2 together with the interval $P_2 Q$ forms a broken line joining P_0 to any point Q of C_{ν_2} and consisting entirely of points of $[P]$.

Since π is not linearly expressible in terms of any subset of $b_{\nu_1}, \cdots, b_{\nu_r}$, the process continues till it shows that P_0 can be joined by a broken line to any point of $[P]$.

14. In like manner, the set of points $[Q]$ interior to and on the boundaries of $C_{\mu_1}, \cdots, C_{\mu_t}$ but not on π, have the property that any two can be joined by a broken line consisting entirely of points Q. Consider now a broken line joining a point P to a point Q. This broken line meets $S^1_{n-1}, \cdots, S^j_{n-1}$ in a finite number of points and intervals. Let these points and the ends of

these intervals be denoted by $A_1, A_2, \cdots, A_\rho$, taken in the sense from P to Q. Since the broken line from P to A_1 is entirely in one of the regions $C_{\nu_1}, \cdots, C_{\nu_\nu}$, A_1 is either in $[P]$ or on π. If A_1 is in $[P]$, it follows in like manner that A_2 is in $[P]$ or on π. Unless one of $A_1, \cdots, A_\rho$ is on π we should thus have Q in $[P]$ or on π. *Hence every broken line joining a P to a Q meets π.*

15. Consider now any point not on π; by definition, it is either a P or a Q. Since π is composed of a finite number of convex $(n-1)$-dimensional regions and their boundaries, any point not on π is interior to an n-dimensional simplex containing no points of π. Any two points interior to such a simplex can be joined by a segment interior to the simplex. Hence all points interior to the simplex are either in $[Q]$ or in $[P]$. *Hence $[P]$ and $[Q]$ are regions and π is the boundary of each region.*

16. Thus we have proved the theorem. *Any $(n-1)$-dimensional polyhedron, π, in an S_n decomposes the S_n into two regions of each of which π is the boundary. Any broken line joining a point of one region to a point of the other contains a point of π.*

Incidentally, we have proved the following corollaries: (1) *Any $(n-1)$-dimensional polyhedron π in an n-space can be regarded as defined in terms of convex $(n-1)$-dimensional polyhedral regions.* (2) *The two regions determined by an $(n-1)$-dimensional polyhedron π in an n-dimensional space are composed each of a finite number of convex polyhedral regions.*

It requires only a slight modification of the argument in § 11 to prove that any $(n-\nu)$-dimensional polyhedron in an S_n is linearly expressible in terms of boundaries of $(n-\nu+1)$-dimensional convex regions. From this theorem it follows easily that the Betti numbers of an S_n are all equal to unity when the class of manifolds with respect to which they are defined consists of polyhedra. For the case in which the manifolds have perfect generality or even when they are composed of analytic pieces no rigorous determination of all the Betti numbers has yet been given.

PRINCETON, N. J.,
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ON CONVERGENCE FACTORS IN DOUBLE SERIES AND THE DOUBLE FOURIER'S SERIES*

BY

CHARLES N. MOORE

Introduction.

It is generally recognized that in order to show that an actual solution of the physical problem is given by the formal series arising in many problems of mathematical physics, it is necessary to apply the theory of convergence factors to these series. Results in this direction of the greatest generality have been obtained by a method originated by FEJÉR,† and applied by him to problems involving the ordinary Fourier's series.

In a previous paper ‡ the writer has applied FEJÉR'S method to problems that involve developments in Bessel's functions. The object of the present paper is to make such an application to problems that involve the development of a function of two variables in a double Fourier's series. Such a discussion naturally involves the consideration of certain general facts in the theory of the summability of double series, a theory that has as yet been scarcely touched on. We have not made, however, any attempt to found a very broad theory of this sort. Such an attempt would have led us too far from the central aim of the paper, and in any case it seems preferable that the theory of summability of double series should be developed at first from the point of view of its applications, and that the study of such a theory from the abstract point of view can be made to better advantage when the facts of widest use in the application of the theory have been brought to light.

The consideration of the summability of double series naturally suggests the consideration of the summability of multiple series of any order, and the study of the summability of the double Fourier's series suggests the study of the summability of the triple Fourier's series and the problems connected with it. Generalizations of the results of the present paper to such cases may be obtained by methods similar to those employed in the subsequent discussion.

* Presented to the Society, April 29, 1911, and December 27, 1911.

† Cf. his well-known memoir, *Untersuchungen über Fouriersche Reihen*, Mathematische Annalen, vol. 58 (1903–04), p. 51.

‡ These Transactions, vol. 10 (1909), p. 391.

§ 1. *Summability of Double Series. Definitions and General Facts.*

We will begin this section by defining the type of summability of double series that we shall be concerned with in this paper. Consider the double series

$$(1)\qquad \begin{aligned} &a_{11}+a_{12}+a_{13}+\cdots\\ &+a_{21}+a_{22}+a_{23}+\cdots\\ &+a_{31}+a_{32}+a_{33}+\cdots\\ &\cdots\cdots\cdots\end{aligned}$$

and form

$$(2)\qquad S_{mn}^{(k)}=\sum_{i=1,\,j=1}^{m,\,n}\frac{\Gamma(k+m-i)}{\Gamma(k)\,\Gamma(m-i+1)}\cdot\frac{\Gamma(k+n-j)}{\Gamma(k)\,\Gamma(n-j+1)}s_{ij},$$

where

$$(3)\qquad s_{ij}=\sum_{p=1,\,q=1}^{i,\,j}a_{pq}.$$

If the quotient

$$(4)\qquad \frac{S_{mn}^{(k)}}{A_{mn}^{(k)}},$$

where

$$(5)\qquad A_{mn}^{(k)}=\frac{\Gamma(m+k)}{\Gamma(k+1)\,\Gamma(m)}\cdot\frac{\Gamma(n+k)}{\Gamma(k+1)\,\Gamma(n)},$$

approaches a limit as m and n become infinite, the double series (1) will be said to be summable (Ck) * and to have a value equal to this limit. This type of summability for double series is seen to be entirely analogous to the type of summability for simple series first considered by CESÀRO.

In the above definition k may be any quantity real or complex, except zero or a negative integer. The definition may at once be extended to the case $k=0$, if we assume that for this value of k the right-hand sides of (2) and (5) have the value that they approach as k approaches zero. It is then seen that summability $(C0)$ is the same as convergence.† In this paper we shall restrict ourselves to the cases where k is a positive integer or zero. The discussion of non-integral orders of summability for double series seems to have no special bearing on the applications we wish to make, and would therefore lead us too far away from the central object of this investigation.‡

We now wish to develop some properties of $S_{mn}^{(k)}$ and $A_{mn}^{(k)}$. It is evident

* We follow here a notation introduced by G. H. HARDY. See Proceedings of the London Mathematical Society, ser. 2, vol. 6 (1908), p. 257.

† We refer here to convergence as defined by PRINGSHEIM. See Münchener Sitzungsberichte, vol. 27 (1897), p. 101, and *Encyklopädie der mathematischen Wissenschaften*, vol. I, A 3, p. 97. Also cf. BROMWICH, *Theory of Infinite Series*, Chap. V.

‡ Non-integral orders of summability of ordinary series have been discussed recently by CHAPMAN, KNOPP, OTTOLENGHI, and M. RIESZ.

from the definitions of these quantities that

$$\sum_{m=1,\,n=1}^{\infty,\,\infty} S_{mn}^{(k)}\, x^m y^n \sim (1-x)^{-k}(1-y)^{-k} \sum_{m=1,\,n=1}^{\infty,\,\infty} s_{mn}\, x^m y^n, \tag{6}$$

$$\sum_{m=1,\,n=1}^{\infty,\,\infty} A_{mn}^{(k)}\, x^m y^n \sim xy\,(1-x)^{-(k+1)}(1-y)^{-(k+1)}, \tag{7}$$

where the sign $\sim$ is used in the sense that if the expressions on the two sides of this sign are expanded in ascending powers of x and y, the coefficients of corresponding terms on the two sides will be equal. The second expression could obviously be written as an equality, since the series on both sides converge absolutely when x and y are less in absolute value than unity. Furthermore, it would not be difficult to show that the same is true of the first relation for the type of series considered in this paper. However, as the expansions are used merely as a device to obtain more simply certain relations between the S's, the s's, the a's, and the A's, relations which depend only on the equations (2), (3), and (5), and which subsist whether the expansions in (6) are convergent or not, it does not seem worth while to consider the question of convergence.

We have at once from (7)

$$\sum_{1,\,1}^{\infty,\,\infty} A_{mn}^{(k)}\, x^m y^n \sim xy\,(1+x+x^2+\cdots)(1+y+y^2+\cdots)(1-x)^{-k}(1-y)^{-k},$$

and on equating the coefficients of $x^m y^n$ on the two sides, we obtain

$$A_{mn}^{(k)} = \sum_{i=1,\,j=1}^{m,\,n} \binom{k+m-i-1}{k-1}\binom{k+n-j-1}{k-1}, \tag{8}$$

so that the denominator in the quotient (4) is equal to the sum of the coefficients of the s's in the numerator. That is to say, we are dealing here with a weighted mean of the sums, just as in CESÀRO's method for ordinary series.

Furthermore, since obviously

$$\sum_{1,\,1}^{\infty,\,\infty} s_{mn}\, x^m y^n \sim (1+x+x^2+\cdots)(1+y+y^2+\cdots) \sum_{1,\,1}^{\infty,\,\infty} a_{mn}\, x^m y^n,$$

it follows from (6) that

$$(1-x)^{k+1}(1-y)^{k+1} \sum_{1,\,1}^{\infty,\,\infty} S_{mn}^{(k)}\, x^m y^n \sim \sum_{1,\,1}^{\infty,\,\infty} a_{mn}\, x^m y^n,$$

whence, equating coefficients on both sides, we obtain

$$a_{mn} = \sum_{i=0,\,j=0}^{k+1,\,k+1} (-1)^i (-1)^j \binom{k+1}{k+1-i}\binom{k+1}{k+1-j} S_{m-i,\,n-j}^{(k)}, \tag{9}$$

where for the sake of uniformity we have set

$$S_{pq}^{(k)} = 0 \qquad (p \text{ or } q \leqq 0). \tag{10}$$

Finally we have from the definition of $A_{mn}^{(k)}$

$$\lim_{m,\,n=\infty} \frac{A_{mn}^{(k)}}{m^k n^k} = \frac{1}{\{\Gamma(k+1)\}^2},$$

and hence

$$A_{mn}^{(k)} < M m^k n^k \qquad (m, n = 1, 2, 3, \cdots), \tag{11}$$

where M is a positive constant.

It is evident that if our definition of summability is to be of wide use, it must be consistent with the more fundamental definition of convergence. That is to say, if a double series is convergent in accordance with the definition of PRINGSHEIM, it should be summable (Ck) for any $k > 0$, and to the same value. We are naturally led to suspect that our definition will satisfy this requirement on account of its analogy to CESÀRO's definition, which is known to satisfy a similar requirement. We shall find that it does so for a very wide class of convergent double series, namely for those for which

$$|s_{mn}| = \left|\frac{S_{mn}^{(0)}}{A_{mn}^{(0)}}\right| < C \qquad (m, n = 1, 2, 3, \cdots), \tag{12}$$

where C is a positive constant. Moreover, when the condition (12) is satisfied for a convergent double series, not only will the ratio $S_{(mn}^{k)}/A_{mn}^{(k)}$ approach the same value as s_{mn} when m and n become infinite, but also it will have in common with s_{mn} the property expressed by (12), namely of remaining less in absolute value than the positive constant C for all values of m and n.

In order to avoid any inconsistency between the theory of summability and that of convergence, or in the theory of summability itself, we will restrict ourselves to the consideration of double series for which

$$\lim_{m,\,n=\infty} \frac{S_{mn}^{(k)}}{A_{mn}^{(k)}} \text{ exists}, \tag{13}$$

and at the same time

$$\left|\frac{S_{mn}^{(k)}}{A_{mn}^{(k)}}\right| < C \qquad (m, n = 1, 2, 3, \cdots), \tag{14}$$

where C is a positive constant, for k equal to zero or some positive integer.

It is possible that some or all of this restriction might be removed. We have not attempted to settle that question in the present paper for two reasons: first, because the series whose summability we proposed to investigate, namely the double Fourier's series, satisfies the restriction; secondly, because in the theorems about convergence factors that we needed for the applications, the same restriction naturally presented itself. However, from the standpoint of a general theory of summability for double series, it is desirable that the necessity of the restriction, or part of the restriction, should be investigated.

If it should turn out that the condition (14) was necessary for consistency in our definitions, it would seem preferable to define the order of summability of a series as the lowest value of k for which both (13) and (14) are satisfied. This, however, would necessitate either imposing the condition (12) in our definition of convergence, or having summability of order zero to agree with convergence only for the class of series for which (12) is satisfied.

Before proceeding to the discussion of the theorem of consistency, we wish to introduce a notation that we shall have occasion to use frequently in this paper, and which we shall use for the first time in the course of establishing the first of two lemmas needed in the proof of the theorem of consistency. The object of this notation is to furnish an abbreviation for certain expressions formed from a double sequence of quantities by selecting a square array of terms from the sequence, attaching to the terms certain binomial coefficients with signs that are alternately plus and minus, and taking the sum of all the resulting quantities. The most general form of such an expression is the following:

$$(15) \qquad \sum_{r=0,\, s=0}^{k+1,\, k+1} (-1)^r (-1)^s \binom{k+1}{k+1-r} \binom{k+1}{k+1-s} f_{i+r,\, j+s}.$$

It will be seen that this expression bears to the double sequence f_{ij} $(i, j = 1, 2, 3, \cdots)$, a relation entirely analogous to that borne to a simple sequence by certain expressions known as $(k+1)$th differences of such a sequence. Hence we may refer to expressions of the form (15) as the $(k+1)$th differences of the corresponding double sequences. Moreover, we will adopt for such expressions a notation analogous to that used for differences of an ordinary sequence; this notation is completely defined by the following expression which represents the quantity (15),

$$(16) \qquad \Delta_{i\,1}^{k+1} f_{ij}.$$

We are now ready to prove the two lemmas needed in the proof of the theorem of consistency.

LEMMA 1. *If we have two double sequences*

$$(17) \qquad a_{mn}, \quad b_{mn} \qquad (m, n = 1, 2, 3, \cdots),$$

such that

$$(a) \qquad \left| \frac{\Delta_1^1 a_{mn}}{\Delta_1^1 b_{mn}} \right| < C \qquad (m, n = 0, 1, 2, 3, \cdots),^*$$

where C is a positive constant,

$$(b) \qquad \Delta_1^1 b_{mn} > 0 \qquad (m, n = 0, 1, 2, 3, \cdots),$$

* In forming $\Delta_1^1 a_{mn}$ and $\Delta_1^1 b_{mn}$ for m or $n = 0$, we put for the sake of uniformity

$$a_{mn} = 0 = b_{mn} \qquad (m \text{ or } n = 0).$$

$$(c) \qquad \lim_{m,\, n=\infty} b_{mn} = \infty ,$$

$$(d) \qquad \lim_{p=\infty} \frac{b_{\mu q}}{b_{pq}} = 0 , \qquad \lim_{q=\infty} \frac{b_{p\nu}}{b_{pq}} = 0 \qquad (\mu ,\ \nu = 1 , 2 , 3 , \cdots),$$

where the limits (d) are approached uniformly, the first for all positive integral values of q, the second for all positive integral values of p,

$$(e) \qquad \lim_{m,\, n=\infty} \frac{\Delta_1^1 a_{mn}}{\Delta_1^1 b_{mn}} \text{ exists;}$$

then we shall have,

$$(18) \qquad \left| \frac{a_{mn}}{b_{mn}} \right| < C \qquad (m , n = 1 , 2 , 3 , \cdots),$$

and furthermore

$$(19) \qquad \lim_{m,\, n=\infty} \frac{a_{mn}}{b_{mn}}$$

*will exist and be equal to the limit in (e).**

We will establish first the inequality (18). We have from conditions (a) and (b)

$$| \Delta_1^1 a_{mn} | < C \Delta_1^1 b_{mn} \qquad (m , n = 1 , 2 , 3 , \cdots),$$

and hence

$$(20) \qquad \sum_{i=0,\, j=0}^{m-1,\, n-1} | \Delta_1^1 a_{ij} | < C \sum_{i=0,\, j=0}^{m-1,\, n-1} \Delta_1^1 b_{ij} .$$

But obviously

$$a_{mn} = \sum_{i=0,\, j=0}^{m-1,\, n-1} \Delta_1^1 a_{ij} , \qquad b_{mn} = \sum_{i=0,\, j=0}^{m-1,\, n-1} \Delta_1^1 b_{ij} ,$$

so that, in view of condition (b) and (20), we have

$$(21) \qquad \left| \frac{a_{mn}}{b_{mn}} \right| \leqq \frac{\sum_{0,\, 0}^{m-1,\, n-1} | \Delta_1^1 a_{ij} |}{\sum_{0,\, 0}^{m-1,\, n-1} \Delta_1^1 b_{ij}} < C \qquad (m , n = 1 , 2 , 3 , \cdots).$$

It now remains to be shown that the limit (19) exists and is equal to the limit in (e).

Since the limit in (e) exists, we can find a μ and ν corresponding to any arbitrarily small, positive quantity ϵ, such that

$$l - \epsilon < \frac{\Delta_1^1 a_{mn}}{\Delta_1^1 b_{mn}} < l + \epsilon \qquad \begin{pmatrix} m \geqq \mu \\ n \geqq \nu \end{pmatrix},$$

where l is the value of the limit in (e). Hence, in view of condition (b), it

* This lemma is a generalization to double sequences of a theorem for ordinary sequences due to Stolz. Cf. Bromwich, *Infinite Series*, p. 378.

follows that

$$(l-\epsilon)\,\Delta_1^1 b_{mn} < \Delta_1^1 a_{mn} < (l+\epsilon)\,\Delta_1^1 b_{mn} \qquad \left(\begin{matrix} m \geqq \mu \\ n \geqq \nu \end{matrix}\right). \tag{22}$$

If we add the inequalities of the form (22) for all pairs of values of m and n such that $\mu \leqq m \leqq p$, $\nu \leqq n \leqq q$, we obtain

$$(l-\epsilon)\,(b_{pq} - b_{\mu q} - b_{p\nu} + b_{\mu\nu}) < a_{pq} - a_{\mu q} - a_{p\nu} + a_{\mu\nu} < (l+\epsilon)\,(b_{pq} - b_{\mu q} - b_{p\nu} + b_{\mu\nu}).$$

If now we take p and q so large that $b_{pq} > 0$, which we can evidently do in view of condition (c), we can divide the last inequality through by b_{pq}. This gives us

$$(l-\epsilon)\left(1 - \frac{b_{\mu q}}{b_{pq}} - \frac{b_{p\nu}}{b_{pq}} + \frac{b_{\mu\nu}}{b_{pq}}\right) < \frac{a_{pq}}{b_{pq}} - \frac{a_{\mu q}}{b_{pq}} - \frac{a_{p\nu}}{b_{pq}} + \frac{a_{\mu\nu}}{b_{pq}} < (l+\epsilon)\left(1 - \frac{b_{\mu q}}{b_{pq}} - \frac{b_{p\nu}}{b_{pq}} + \frac{b_{\mu\nu}}{b_{pq}}\right). \tag{23}$$

But from conditions (c) and (d) it follows that the limit

$$\lim_{p,\,q=\infty}\left[1 - \frac{b_{\mu q}}{b_{pq}} - \frac{b_{p\nu}}{b_{pq}} + \frac{b_{\mu\nu}}{b_{pq}}\right]$$

exists and is equal to unity. Moreover, since

$$-\frac{a_{\mu q}}{b_{pq}} - \frac{a_{p\nu}}{b_{pq}} + \frac{a_{\mu\nu}}{b_{pq}} = -\frac{a_{\mu q}}{b_{\mu q}} \cdot \frac{b_{\mu q}}{b_{pq}} - \frac{a_{p\nu}}{b_{p\nu}} \cdot \frac{b_{p\nu}}{b_{pq}} + \frac{a_{\mu\nu}}{b_{pq}},$$

it follows from conditions (c) and (d) and the inequality (21) that the limit

$$\lim_{p,\,q=\infty}\left[-\frac{a_{\mu\nu}}{b_{pq}} - \frac{a_{p\nu}}{b_{pq}} + \frac{a_{\mu\nu}}{b_{pq}}\right]$$

exists and is equal to zero. Hence we obtain from (23)

$$l - \epsilon \leqq \underline{\lim}\,\frac{a_{pq}}{b_{pq}} \leqq \overline{\lim}\,\frac{a_{pq}}{b_{pq}} \leqq l + \epsilon.$$

Since ϵ is an arbitrarily small positive quantity, it follows at once from the last inequality that

$$\underline{\lim}\,\frac{a_{pq}}{b_{pq}} = \overline{\lim}\,\frac{a_{pq}}{b_{pq}} = l.$$

Consequently the limit (19) exists and is equal to l, the value of the limit in (e). Our lemma is therefore proved.

LEMMA 2. *If the double series* (1) *is summable* (Ck), *and furthermore*

$$\left|\frac{S_{mn}^{(k)}}{A_{mn}^{(k)}}\right| < C \qquad (m, n = 1, 2, 3, \cdots), \tag{24}$$

where C is a positive constant, the series will be summable $(C\overline{k+1})$ to the same value, and furthermore we shall have

$$\left|\frac{S_{mn}^{(k+1)}}{A_{mn}^{(k+1)}}\right| < C \qquad (m, n = 1, 2, 3, \cdots). \tag{25}$$

From relations (6) and (7) it follows at once that

$$\sum_{1,1}^{\infty,\infty} S_{mn}^{(k+1)} x^m y^n \sim (1+x+x^2+\cdots)(1+y+y^2+\cdots)\sum_{1,1}^{\infty,\infty} S_{mn}^{(k)} x^m y^n,$$

$$\sum_{1,1}^{\infty,\infty} A_{mn}^{(k+1)} x^m y^n \sim (1+x+x^2+\cdots)(1+y+y^2+\cdots)\sum_{1,1}^{\infty,\infty} A_{mn}^{(k)} x^m y^n,$$

whence, on equating coefficients of corresponding terms, we have

$$S_{mn}^{(k+1)} = \sum_{1,1}^{m,n} S_{ij}^{(k)}, \qquad A_{mn}^{(k+1)} = \sum_{1,1}^{m,n} A_{ij}^{(k)}. \tag{26}$$

From these two equations we obtain

$$S_{mn}^{(k)} = \Delta_1^1 S_{m-1,n-1}^{(k+1)}, \qquad A_{m,n}^{(k)} = \Delta_1^1 A_{m-1,n-1}^{(k+1)}, \tag{27}$$

where for the sake of uniformity we make use again of the notation (10).

We are now ready to apply Lemma 1, taking for the first sequence in (17) the quantities $S_{mn}^{(k+1)}$, and for the second sequence the quantities $A_{mn}^{(k+1)}$. It follows at once from condition (24) of the present lemma, and equation (27), that condition (a) of Lemma 1 is satisfied; that condition (b) is satisfied is an immediate consequence of the second equation in (27) and the definition of $A_{mn}^{(k)}$ (see equation (5)). It also follows without difficulty from the same definition that (c) and (d) of Lemma 1 are satisfied, and that condition (e) is satisfied follows from the hypothesis of the present lemma that the double series (1) is summable (Ck).

Since all the conditions of Lemma 1 are satisfied by the quantities under consideration, it follows from that lemma that

$$\lim_{m,n=\infty} \frac{S_{mn}^{(k+1)}}{A_{mn}^{(k+1)}}$$

exists and is equal to

$$\lim_{m,n=\infty} \frac{\Delta_1^1 S_{mn}^{(k+1)}}{\Delta_1^1 A_{mn}^{(k+1)}} = \lim_{m,n=\infty} \frac{S_{mn}^{(k)}}{A_{mn}^{(k)}},$$

that is that the series (1) is summable $(C\overline{k+1})$ and to the same value to which it is summable (Ck).

It also follows from Lemma 1 that

$$\left|\frac{S_{mn}^{(k+1)}}{A_{mn}^{(k+1)}}\right| < C \qquad (m, n = 1, 2, 3, \cdots).$$

The present lemma is, therefore, completely established.

We are now prepared to establish the theorem of consistency which may be stated as follows:

THEOREM I. *If the series* (1) *is summable* (Ck), *where* k *is zero or any positive integer, and if furthermore*

$$\left|\frac{S_{mn}^{(k)}}{A_{mn}^{(k)}}\right| < C \qquad (m, n = 1, 2, 3, \cdots),$$

where C *is a positive constant, then the series will be summable* (Ck'), *where* k' *is any integer greater than* k, *and to the same value, and furthermore we shall have*

$$\left|\frac{S_{mn}^{(k')}}{A_{mn}^{(k')}}\right| < C \qquad (m, n = 1, 2, 3, \cdots).$$

If $k' - k = 1$, this theorem reduces to Lemma 2. If $k' - k = r > 1$, the theorem follows at once from r successive applications of Lemma 2.

Before closing this section we wish to define uniform convergence and uniform summability of double series, and point out a few facts in this connection. If the terms of the series (1) are functions of x and y throughout a certain region R of the x, y-plane, and furthermore s_{mn}, defined by equation (3), approaches a limit uniformly for all values of x and y in R as m and n become infinite, we shall say that the double series (1) is uniformly convergent. The definition for any other number of variables is entirely analogous. Similarly we define the series to be uniformly summable (Ck) in R if the quotient (4) approaches a limit uniformly for all values of x and y in R, as m and n become infinite.

It is easy to show that if the terms of a double series are continuous throughout a region in which the series is uniformly convergent or uniformly summable, the function which the series represents will be continuous. The proof is entirely analogous to the well-known proof for simple series that a uniformly convergent series of continuous functions defines a continuous function. Moreover, it is easy to establish a test for the uniform convergence of a double series analogous to the well-known test of Weierstrass for the uniform convergence of a simple series, namely to show that if the general term of a double series is less in absolute value than the general term of a convergent double series of positive constants, throughout a certain region, the given double series will converge uniformly throughout that region.

§ 2. *Convergence Factors in Double Series.*

Before entering on the subject matter of this section we wish to define one further notation that we introduce for the sake of brevity. In analogy with the abbreviated notation introduced to represent the expression (15), we will set

$$(28)\quad \Delta_{k+1,q}^{k+1,p} f_{ij} = \sum_{r=0,\,s=0}^{k+1-q,\,k+1-p} (-1)^r (-1)^s \binom{k+1}{k+1-r}\binom{k+1}{k+1-s} f_{i+r,\,j+s}.$$

It will be observed that the expression on the right-hand side of (28) is equal to the expression (15) with certain of its terms suppressed. The number of terms suppressed and the ones which are suppressed, are indicated by the second upper and lower indices added to the expression (16) to form the left-hand side of (28). For the cases in which p or q is equal to $k+1$ we shall adopt a simpler way of writing the left-hand side of (28)*; thus we shall set

$$\Delta_{k+1,q}^{k+1;\,k+1} f_{ij} = \Delta_{k+1,\,q} f_{ij}, \qquad \Delta_{k+1,k+1}^{k+1;\,p} f_{ij} = \Delta^{k+1,p} f_{ij}. \tag{29}$$

This notation having been explained, we are ready to prove the theorems on convergence factors that we shall consider in this paper. We begin by considering the case of convergence factors in double series that are summable $(C1)$. The results that we shall establish for this case are special cases of the results that we shall obtain later on for the case of convergence factors in series summable (Ck). However, as these special cases are the ones that we shall make use of in the applications of this paper, it seems worth while to prove them separately, especially as the proof is considerably simpler than the proof of the general case.

LEMMA 3. *If the double series* (1) *is summable* $(C1)$, *and moreover*

$$\left|\frac{S_{mn}^{(1)}}{A_{mn}^{(1)}}\right| < C \qquad (m, n = 1, 2, 3, \cdots), \tag{30}$$

where C is a positive constant, then for all positive values of α and β the series

$$\sum_{m=1,\,n=1}^{\infty,\,\infty} a_{mn} f_{mn}(\alpha, \beta) \tag{31}$$

will converge and have the same value as the series

$$\sum_{i=1,\,j=1}^{\infty,\,\infty} S_{ij}^{(1)} \Delta_2^2 f_{ij}(\alpha, \beta), \tag{32}$$

which will also be convergent, provided the convergence factors $f_{ij}(\alpha, \beta)$ satisfy the following conditions:

$$(a) \qquad \sum_{i=1,\,j=1}^{\infty,\,\infty} ij\,|\Delta_2^2 f_{ij}(\alpha, \beta)| < K, \text{ a positive constant} \qquad (\alpha, \beta > 0),$$

$$(b) \qquad \lim_{m=\infty} m \sum_{j=1}^{\infty} j\,|f_{mj}(\alpha, \beta)| = 0 \qquad (\alpha, \beta > 0),$$

$$(c) \qquad \lim_{n=\infty} n \sum_{i=1}^{\infty} i\,|f_{in}(\alpha, \beta)| = 0 \qquad (\alpha, \beta > 0),$$

$$(d) \qquad \lim_{m,\,n=\infty} [mn f_{mn}(\alpha, \beta)] = 0 \qquad (\alpha, \beta > 0),$$

* If either p or q is equal to zero, the zero value will be omitted.

it being understood of course that the limiting processes indicated in conditions (a), (b), (c), and (d), all have a meaning.

Substituting in the expression

$$\sum_{1,1}^{m,n} a_{ij} f_{ij} \tag{33}$$

the value of a_{ij} given by putting $k = 1$ in (9), and rearranging the terms according to suffixes of S, we obtain the expression *

$$\begin{aligned}
&\sum_{1,1}^{m-2,n-2} S_{ij}^{(1)} \Delta_2^2 f_{ij} + \sum_{i=1}^{i=m-2} S_{i,n-1}^{(1)} \Delta_2^{2,1} f_{i,n-1} \\
&\quad + \sum_{i=1}^{i=m-2} S_{in}^{(i)} \Delta_2 f_{in} + \sum_{j=1}^{j=n-2} S_{m-1,j}^{(1)} \Delta_{2,1}^2 f_{m-1,j} \\
&\quad + \sum_{j=1}^{j=n-2} S_{mj}^{(1)} \Delta^2 f_{mj} + S_{m-1,n-1}^{(1)} \Delta_{2,1}^{2,1} f_{m-1,n-1} \\
&\quad + S_{mn}^{(1)} f_{mn} + S_{m-1,n}^{(1)} \Delta_{2,1} f_{m-1,n} + S_{m,n-1}^{(1)} \Delta^{2,1} f_{m,n-1}.
\end{aligned} \tag{34}$$

We shall show that for positive values of α and β, each term of the expression (34) approaches a limit as m and n become infinite, and that every term but the first approches the limit zero. From this it will follow that the series (31) converges for all positive values of α and β and to the same limit as the series

* The simplest way to see that the expression (34) is identically equal to the expression (33) is the following: On expanding the terms of (33), we observe that each term of the double summation in (34) is obtained by selecting according to a fixed rule one term out of each of the expansions of nine terms of (33), these terms of (33) forming a square array whose upper left hand corner is the term with subscripts p and q, in the case where we are forming the term in the double summation of (34) whose subscripts are p and q. It is easily seen in this way that all the terms in the double summation of (34) can be obtained from the expansions of the terms of (33), and that this process uses up all the terms in the expansions of the $(m-2)(n-2)$ terms of (33) lying in a rectangle $(m-2)$ terms high and $(n-2)$ terms broad whose upper left-hand corner is the first term of (33), and some of the terms in the expansions of each of the other terms. It remains to verify the fact that the remaining terms in the expansions of the terms of (33) are identically equal to the terms in (34) after the first. This is easily seen by selecting terms from the expansions according to the same rule by which the terms of the double summation of (34) were obtained. It is obvious that such a process will give us *incomplete* terms of the form of the terms in this double summation, and that the degree of incompleteness depends on the position with regard to the edges of the rectangle of mn terms formed by (33), of the term in whose expansion we select the first term to form the term of (34). The terms of (34) whose first terms are taken from one of the terms in the first $(m-2)$ terms of the $(n-1)$th column in the double summation (33) form the first single summation in (34); the terms of (34) whose first terms are taken from one of the first $(m-2)$ terms of the nth column of (33) form the second single summation of (34), and so on. By accounting for all terms whose first term comes from the expansion of one of the terms of (33) that lie in the last two rows, the last two columns, or both, we get all the terms of (34) and we use up all the terms in the expansions of the terms of (33).

An entirely similar process will enable us to make the more complicated reduction required later on to reduce (33) to (49).

(32) whose convergence will be established in proving that the first term in (34) approaches a limit as m and n become infinite. Thus our lemma will be completely established.

Consider first the first term in (34). This term is the sum of the $(m-2)(n-2)$ terms of the series (32) contained in a rectangle $(m-2)$ terms high and $(n-2)$ terms broad, taken from the upper left-hand corner of that series. In view of (30), we have for the general term of (32)

$$|S^{(1)}_{mn}\Delta_2^2 f_{mn}| < Cmn\,|\Delta_1^2 f_{mn}| \quad \begin{pmatrix} m, n = 1, 2, 3, \cdots \\ \alpha, \beta > 0 \end{pmatrix}. \tag{35}$$

But it follows from condition (a) that the series whose general term is the expression on the right-hand side of (35) is convergent. Hence the series (32) is absolutely convergent, and consequently the first term in (34) ap-approaches a limit as m and n become infinite. That the sixth, seventh, eighth, and ninth terms approach zero as m and n become infinite is an immediate consequence of conditions (30) and (d). It remains to show that the same is true of the second, third, fourth, and fifth terms.

The second term of (34) is in absolute value less than or equal to

$$\sum_{i=1}^{m-2}|S^{(1)}_{i,n-1}f_{i,n-1}| + 2\sum_{i=1}^{m-2}|S^{(1)}_{i,n-1}f_{i+1,n-1}| + \sum_{i=1}^{m-2}|S^{(1)}_{i,n-1}f_{i+2,n-1}|$$
$$+ 2\sum_{i=1}^{m-2}|S^{(1)}_{i,n-1}f_{in}| + 4\sum_{i=1}^{m-2}|S^{(1)}_{i,n-1}f_{i+1,n}| + 2\sum_{i=1}^{m-2}|S^{(1)}_{i,n-1}f_{i+2,n}|,$$

which in view of (30) is in turn less than

$$C(n-1)\sum_{i=1}^{m-2} i\,|f_{i,n-1}| + 2C(n-1)\sum_{i=1}^{m-2} i\,|f_{i+1,n-1}| + C(n-1)\sum_{i=1}^{m-2} i\,|f_{i+2,n-1}|$$
$$+ 2C(n-1)\sum_{i=1}^{m-2} i\,|f_{in}| + 4C(n-1)\sum_{i=1}^{m-2} i\,|f_{i+1,n}| + 2C(n-1)\sum_{i=1}^{m-2} i\,|f_{i+2,n}|$$
$$< 4C(n-1)\sum_{i=1}^{m} i\,|f_{i,n-1}| + 8C(n-1)\sum_{i=1}^{m} i\,|f_{in}|.$$

It follows at once from condition (c) that the expression on the right-hand side of the last inequality approaches zero as m and n become infinite. Hence the second term of (34), which is always less in absolute value than this expression, must also approach zero as a limit.

Consider now the third term of (34). It is obviously in absolute value less than or equal to

$$\sum_{i=1}^{m-2}|S^{(1)}_{in}f_{in}| + 2\sum_{i=1}^{m-2}|S^{(1)}_{in}f_{i+1,n}| + \sum_{i=1}^{m-2}|S^{(1)}_{in}f_{i+2,n}|,$$

which in view of (30) is in turn less than

$$Cn\sum_{i=1}^{m-2} i\,|f_{in}| + 2Cn\sum_{i=1}^{m-2} i\,|f_{i+1,\,n}| + Cn\sum_{i=1}^{m-2} i\,|f_{i+2,\,n}| < 4Cn\sum_{i=1}^{m} i\,|f_{in}|.$$

It follows at once from condition (*c*) that the expression on the right-hand side of the last inequality approaches zero as m and n become infinite. Hence the third term of (34), which is always less in absolute value than this expression, must also approach zero as a limit.

That the fourth and fifth terms of the expression (34) approach zero as a limit as m and n become infinite, may be shown by a discussion analogous to that used in the case of the second and third terms, the principal difference being that condition (*b*) replaces condition (*c*).

Thus we see that every term in (34) approaches a limit as m and n become infinite and that every term but the first approaches the limit zero. Hence, as we have pointed out before, our lemma is completely established.

We are now ready to prove the following theorem:

THEOREM II.* *If the double series* (1) *satisfies the conditions of Lemma* 3, *and the convergence factors* $f_{ij}(\alpha, \beta)$ *satisfy the conditions of that lemma and the following further conditions*

(*e*) $\qquad f_{ij}(\alpha, \beta)$ *is continuous is* α *and* β $\quad \begin{pmatrix} i, j = 1, 2, 3, \cdots \\ \alpha, \beta > 0 \end{pmatrix}$,

(*f*) $\qquad \lim_{\alpha, \beta = +0} [f_{ij}(\alpha, \beta)] = f_{ij}(0, 0) = 1 \quad (i, j = 1, 2, 3, \cdots)$,

then the series (31) *will define a function of* α *and* β, $F(\alpha, \beta)$, *that is continuous for all positive values of* α *and* β, *and for which*

$$\lim_{\alpha, \beta = +0} [F(\alpha, \beta)] = S, \tag{36}$$

where S *is the value of the series* (1).

We know from Lemma 3 that under the conditions of that lemma the series (31) converges to the same value as the series (32) for all positive values of α and β. Hence if we can show that under the conditions of the present theorem the series (32) defines a function of α and β that is continuous for all positive values of α and β and for which (36) is true, our theorem will be proved.

If we set

$$\frac{S_{mn}^{(1)}}{mn} = \sigma_{mn}, \tag{37}$$

* This theorem includes as a special case a theorem proved by BROMWICH and HARDY. Cf. Proceedings of the London Mathematical Society, 2d ser., vol. 2 (1905), p. 161. In the same paper the authors consider series summable by higher means, but the means used are the analogues of HÖLDER's means for simple series and not of CESÀRO's.

we shall have

$$\lim_{m,\, n=\infty} [\sigma_{mn}] = S.$$

Hence if we put

(38) $$\sigma_{mn} = S + \epsilon_{mn},$$

it will follow that

(39) $$\lim_{m,\, n=\infty} [\epsilon_{mn}] = 0.$$

By means of (37) and (38) we can reduce the series (32) to the form

(40) $$S \sum_{1,1}^{\infty,\infty} ij\Delta_2^2 f_{ij}(\alpha, \beta) + \sum_{1,1}^{\infty,\infty} ij\epsilon_{ij}\Delta_2^2 f_{ij}(\alpha, \beta).$$

We can evaluate the series in the first term of (40) by applying Lemma 3 to the double series in which

(41) $$a_{11} = 1, \qquad a_{ij} = 0 \qquad (i \text{ or } j > 1).$$

This series is obviously convergent, and moreover satisfies the restriction (12); hence by Theorem I it is summable ($C1$). It therefore follows from Lemma 3 that for this series

(42) $$\sum_{1,1}^{\infty,\infty} S_{ij}^{(1)}\Delta_2^2 f_{ij}(\alpha, \beta) = \sum_{1,1}^{\infty,\infty} a_{ij} f_{ij}(\alpha, \beta) \qquad (\alpha, \beta > 0).$$

Making use of the values of the terms of the series (41) and the definition (2) of $S_{mn}^{(1)}$, we have for this series

(43) $$S_{ij}^{(1)} = ij \qquad (i, j = 1, 2, 3, \cdots),$$

by means of which equation (42) reduces to the form

(44) $$\sum_{1,1}^{\infty,\infty} ij\Delta_2^2 f_{ij}(\alpha, \beta) = f_{11}(\alpha, \beta) \qquad (\alpha, \beta > 0).$$

From (44) it follows that the first term in (40) is equal to

$$Sf_{11}(\alpha, \beta).$$

This quantity is a continuous function of α and β for all positive values of α and β in view of condition (e), and from condition (f) it is seen that it approaches S as α and β approach $+0$. Since (40) is only another way of writing the series (32), it only remains to show that the second term of (40) is continuous for all positive values of α and β and that it approaches zero as α and β approach $+0$. We will show this by proving that the terms of the series in the second term of (40) are continuous in the region

(45) $$\begin{pmatrix} \alpha > 0, \ \beta > 0 \\ \alpha = 0 = \beta \end{pmatrix},$$

that the series is uniformly convergent in that region, and that its value is zero for $\alpha = 0 = \beta$. It will then follow from one of the theorems referred to at the close of § 1, namely from the theorem that a double series which is uniformly convergent in a region in which its terms are continuous, defines there a continuous function, that the second term of (40) has the desired properties. Thus our theorem will be completely established.

That the terms of the series in the second term of (40) are continuous in (45) follows at once from conditions (*e*) and (*f*). It remains to be shown then that the series is uniformly convergent in that region, and that it converges to zero for $\alpha = 0 = \beta$.

For m and n sufficiently large $|\epsilon_{ij}|$ becomes less than an arbitrarily small quantity ϵ by (39), and hence we have

$$(46) \qquad \left| \sum_{m,n}^{p,q} ij\epsilon_{ij} \Delta_2^2 f_{ij}(\alpha, \beta) \right| \leqq \epsilon \sum_{m,n}^{p,q} ij \left| \Delta_2^2 f_{ij}(\alpha, \beta) \right| < K\epsilon \quad (\alpha, \beta > 0),$$

by (*a*), Lemma 3.

For $\alpha = 0 = \beta$ each term of the series in the second term of (40) is identically zero, and hence the inequality (46) holds for those values also. Therefore the series just referred to converges uniformly throughout the region (45) and, moreover, it converges to zero for $\alpha = 0 = \beta$, since each term is identically zero for such values. We saw above that the terms of the series were continuous in (45), and hence, as we pointed out before, our theorem is completely established.

In Lemma 3 and Theorem II we have assumed that the series in which the convergence factors were introduced had constant terms. This limitation was made in order to avoid increasing the complexity of the formulæ and also in order to keep more clearly in mind the central fact of the theorem. Moreover, the extension to the case in which the terms of the series are functions of one or more variables is easily made. The most important fact in this connection is stated in the following corollary, where for the sake of definiteness and because that is the case of most interest in the applications, we restrict ourselves to series whose terms are functions of two variables.

COROLLARY. *If the series*

$$\sum_{1,1}^{\infty,\infty} a_{ij}(x, y)$$

is uniformly summable $(C1)$ *to the function* $f(x, y)$ *throughout a certain region* R, *and moreover*

$$\left| \frac{S_{mn}^{(1)}(x, y)}{mn} \right| < C \qquad (m, n = 1, 2, 3, \cdots),$$

where C is a positive constant, then the series

$$\sum_{1,1}^{\infty,\infty} a_{ij}(x, y) f_{ij}(\alpha, \beta)$$

will converge for all positive values of α and β and all values of x and y in R, and its value $F(x, y, \alpha, \beta)$ will approach $f(x, y)$ uniformly for all values of x and y in R, as α and β approach $+0$, provided the convergence factors $f_{ij}(\alpha, \beta)$ satisfy the conditions of Theorem II.

The proof of this corollary requires only slight modifications in the reasoning of Lemma 3 and Theorem II.

We will next consider the case of convergence factors in series that are summable (Ck), where k is zero or any positive integer. As in the special case just treated, we begin by proving a lemma.

LEMMA 4. *If the double series* (1) *is summable (Ck), and moreover*

$$\left|\frac{S_{mn}^{(k)}}{A_{mn}^{(k)}}\right| < C \qquad (m, n = 1, 2, 3, \cdots), \tag{47}$$

where C is a positive constant, then the series (31) *will converge for all positive values of α and β, and will have for such values of α and β the same value as the series*

$$\sum_{i=1,\, j=1}^{\infty,\infty} S_{ij}^{(k)} \Delta_{k+1}^{k+1} f_{ij}(\alpha, \beta), \tag{48}$$

which will also converge, provided the convergence factors $f_{ij}(\alpha, \beta)$ satisfy the following conditions

$$(a) \qquad \sum_{i=1,\, j=1}^{\infty,\infty} i^k j^k \,|\, \Delta_{k+1}^{k+1} f_{ij}(\alpha, \beta) \,| < K \text{ a positive constant} \qquad (\alpha, \beta > 0),$$

$$(b) \qquad \lim_{m=\infty} m^k \sum_{j=1}^{\infty} j^k \,|\, f_{mj}(\alpha, \beta) \,| = 0 \qquad (\alpha, \beta > 0),$$

$$(c) \qquad \lim_{n=\infty} n^k \sum_{i=1}^{\infty} i^k \,|\, f_{in}(\alpha, \beta) \,| = 0 \qquad (\alpha, \beta > 0),$$

$$(d) \qquad \lim_{m,\, n=\infty} [\, m^k n^k f_{mn}(\alpha, \beta) \,] = 0 \qquad (\alpha, \beta > 0),$$

it being understood that the limiting processes indicated in conditions (a), (b), (c), and (d), all have a meaning.

Substituting in (33) the value of a_{ij} given by (9) and rearranging the terms according to suffixes of S, we obtain the expression *

* For the method of making the required computation see the footnote to Lemma 3.

$$
\begin{aligned}
&\sum_{1,\,1}^{m-k-1,\,n-k-1} S_{ij}^{(k)}\ \Delta_{k+1}^{k+1} f_{ij} + \sum_{i=1,\,p=0}^{m-k-1,\,k} S_{i,\,n-k+p}^{(k)}\ \Delta_{k+1}^{k+1,\,p+1} f_{i,\,n-k+p} \\
&\qquad + \sum_{p=0,\,j=1}^{k,\,n-k-1} S_{m-k+p,\,j}^{(k)}\ \Delta_{k+1,\,p+1}^{k+1} f_{m-k+p,\,j} \\
&\qquad + \sum_{p=0}^{k} S_{m-k+p,\,n-k+p}^{(k)}\ \Delta_{k+1,\,p+1}^{k+1,\,p+1} f_{m-k+p,\,n-k+p} \\
&\qquad + \sum_{p=0}^{k-1}\left[\sum_{q=p+1}^{k} S_{m-k+p,\,n-k+q}^{(k)}\ \Delta_{k+1,\,p+1}^{k+1,\,q+1} f_{m-k+p,\,n-k+q}\right] \\
&\qquad + \sum_{p=0}^{k-1}\left[\sum_{q=p+1}^{k} S_{m-k+q,\,n-k+p}^{(k)}\ \Delta_{k+1,\,q+1}^{k+1,\,p+1} f_{m-k+q,\,n-k+p}\right].
\end{aligned}
\tag{49}
$$

We shall establish our lemma by showing that for $\alpha > 0$, $\beta > 0$ each of the terms in the above expression approaches a limit as m and n become infinite, and that every term but the first approaches the limit zero. Thus it will follow that the series (31) converges for $\alpha > 0$, $\beta > 0$, and to the same limit as the series (48), whose convergence will be established in proving that the first term of (49) approaches a limit as m and n become infinite.

Consider first the first term in (49). This term is the sum of the $(m-k-1)(n-k-1)$ terms of the series (48) contained in a rectangle $(m-k-1)$ terms high and $(n-k-1)$ terms broad, taken from the upper left-hand corner of that series. In view of (47) and (11), we have for the general term of (48)

$$
|\, S_{mn}^{(k)}\ \Delta_{k+1}^{k+1} f_{mn}\,| < CM\, m^k\, n^k\, |\, \Delta_{k+1}^{k+1} f_{mn}\,| \qquad (\alpha,\ \beta > 0). \tag{50}
$$

But it follows from condition (a) that the series whose general term is the expression on the right-hand side of (50) is convergent. Hence the series (48) is absolutely convergent, and consequently the first term of (49) approaches a limit as m and n become infinite, this limit being the value of the convergent series (48).

It remains now to be shown that the other terms in (49) approach the limit zero as m and n become infinite. The fourth, fifth, and sixth terms of this expression each consist of a fixed number of terms, that is a number dependent on k but not on m and n. Moreover each term is of the form

$$
N S_{ij}\, f_{i+r,\ j+s}, \tag{51}
$$

where N is a coefficient that is never greater in absolute value than the largest coefficient in the expansion of $(a+b)^{k+1}(c+d)^{k+1}$, i and j are positive integers that become infinite with m and n, and r and s are integers, each of which may have any one of the values $0, 1, 2, \cdots, k$.

It follows easily from conditions (47) and (*d*) that an expression of the form (51) approaches zero as a limit as m and n become infinite. For combining (47) with (11) we have

$$|NS_{ij}f_{i+r,\,j+s}| < NCMij\,|f_{i+r,\,j+s}| \leqq NCM\,(i+r)\,(j+s)\,|f_{i+r,\,j+s}|.$$

From condition (*d*) and the fact that i and j are integers that become infinite with m and n, it follows at once that the expression on the right-hand side of the last inequality approaches zero as m and n become infinite. Hence the expression on the left-hand side, and consequently the expression (51) approach zero as m and n become infinite. As the fourth, fifth, and sixth terms of (49) each consist of a fixed number of terms of the form (51), it follows that they also approach zero as m and n become infinite.

We have still to consider the second and third terms of (49). Here the number of terms in the summations is not fixed but varies with m and n, and we have to make use of conditions (*b*) and (*c*) in order to show that the quantities we are discussing approach zero as m and n become infinite.

For the second term of (49) we have, in view of (47) and (11),

$$\left|\sum_{i=1,\,p=0}^{m-k-1,\,k} S_{i,\,n-k+p}^{(k)}\,\Delta_{k+1}^{k+1,\,p+1}f_{i,\,n-k+p}\right| \leqq \sum_{i=1,\,p=0}^{m-k-1,\,k} |S_{i,\,n-k+p}^{(k)}|\cdot|\Delta_{k+1}^{k+1,\,p+1}f_{i,\,n-k+p}| \tag{52}$$

$$< CM\sum_{i=1}^{m-k-1} i^k\left[\sum_{p=0}^{k}(n-k+p)^k\,|\,\Delta_{k+1}^{k+1,\,p+1}f_{i,\,n-k+p}\,|\right].$$

But, since

$$1+(k+1)+\frac{(k+1)\,k}{2!}+\cdots+(k+1)+1$$
$$+(k+1)+(k+1)^2+\frac{(k+1)^2\,k}{2!}+\cdots+(k+1)^2+(k+1)$$
$$+\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$+(k+1)+(k+1)^2+\frac{(k+1)^2\,k}{2!}+\cdots+(k+1)^2+(k+1)$$
$$+1+(k+1)+\frac{(k+1)\,k}{2!}+\cdots+(k+1)+1=(1+1)^{k+1}(1+1)^{k+1}=4^{k+1},$$

it is not difficult to see, if we write out the terms of the various expressions

$$\Delta_{k+1}^{k+1,\,p+1}f_{i,\,n-k+p}$$

and make use of the principle that the absolute value of the sum of several quantities is less than or equal to the sum of the absolute values, that

$$\sum_{p=0}^{k}(n-k+p)^k\,|\,\Delta_{k+1}^{k+1,\,p+1}f_{i,\,n-k+p}\,| \tag{53}$$
$$< 4^{k+1}\sum_{p=0}^{k}(n-k+p)^k\left\{\sum_{q=0}^{k+1}|\,f_{i+q,\,n-k+p}\,|\right\}.$$

Combining (52) and (53), we obtain

$$\left|\sum_{i=1,\,p=0}^{m-k-1,\,k} S^{(k)}_{i,\,n-k+p}\,\Delta^{k+1,\,p+1}_{k+1}\,f_{i,\,n-k+p}\right| < 4^{k+1}CM\sum_{p=0}^{k}(n-k+p)^k\left[\sum_{i=1}^{m-k-1} i^k\left\{\sum_{q=0}^{k+1}|f_{i+q,\,n-k+p}|\right\}\right]. \tag{54}$$

But

$$\sum_{i=1}^{m-k-1} i^k\left\{\sum_{q=0}^{k+1}|f_{i+q,\,n-k+p}|\right\} < (k+2)\sum_{i=1}^{m} i^k\,|f_{i,\,n-k+p}|, \tag{55}$$

and hence we have finally, from (54) and (55),

$$\left|\sum_{i=1,\,p=0}^{m-k-1,\,k} S^{(k)}_{i,\,n-k+p}\,\Delta^{k+1,\,p+1}_{k+1}\,f_{i,\,n-k+p}\right| < (k+2)\,4^{k+1}CM\sum_{p=0}^{k}(n-k+p)^k\left\{\sum_{i=1}^{m} i^k\,|f_{i,\,n-k+p}|\right\}. \tag{56}$$

It follows without difficulty from condition (*c*) that the right-hand member of (56) approaches zero as m and n become infinite. Consequently the left-hand member, that is to say the second term of (49), approaches zero as m and n become infinite.

By a discussion precisely analogous to the above we may obtain for the third term of (49) the inequality

$$\left|\sum_{p=0,\,j=1}^{k,\,m-k-1} S^{(k)}_{m-k+p,\,j}\,\Delta^{k+1,\,p+1}_{k+1}\,f_{m-k+p,\,j}\right| < (k+2)\,4^{k+1}CM\sum_{p=0}^{k}(m-k+p)^k\left\{\sum_{j=1}^{n} j^k\,|f_{m-k+p,\,j}|\right\}. \tag{57}$$

By means of this inequality and condition (*b*) we may infer at once that the third term of (49) approaches zero as m and n become infinite.

We have now shown that each term in (49) approaches a limit as m and n become infinite, and that every term but the first approaches the limit zero. Hence, as we have pointed out before, our lemma is completely established.

With the aid of Lemma 4 we can prove the following theorem:

THEOREM III.* *If the double series* (1) *satisfies the conditions of Lemma* 4, *and the convergence factors* $f_{ij}(\alpha, \beta)$ *satisfy the conditions of that lemma and the following further conditions*

(*e*) $f_{ij}(\alpha, \beta)$ *is continuous in* α *and* β $\left(\begin{matrix} i, j = 1, 2, 3, \cdots \\ \alpha, \beta > 0 \end{matrix}\right)$,

* Theorem III is a generalization to the case of double series of a theorem for simple series due to BROMWICH. Cf. Mathematische Annalen, vol. 65 (1908), p. 358. Recently BROMWICH's theorem has been generalized to the case of simple series whose order of summability is non-integral by CHAPMAN and by OTTOLENGHI. Cf. Proceedings of the London Mathematical Society, ser. 2, vol. 9 (1911), p. 382, and Giornale di Matematiche di Battaglini, vol. 49 (1911), p. 245.

$$(f) \qquad \lim_{\alpha,\beta=+0} [f_{ij}(\alpha, \beta)] = f_{ij}(0, 0) = 1 \qquad (i, j = 1, 2, 3, \cdots),$$

then the series (31) *will define a function of* α *and* β, $F(\alpha, \beta)$, *that is continuous for all positive values of* α *and* β, *and for which*

$$\lim_{\alpha,\beta=+0} [F(\alpha, \beta)] = S,$$

where S *is the value of the series* (1).

The above theorem is proved with the aid of Lemma 4 in almost precisely the same manner that Theorem II was proved with the aid of Lemma 3. Practically the only changes in the wording of the proof which are not merely those of notation consist in the substitution for the equality (43), of the equality

$$S_{ij}^{(k)} = \sum_{p=1,\, q=1}^{i,\, j} \binom{k+i-p-1}{k-1}\binom{k+i-q-1}{k-1} = A_{ij}^{(k)} \qquad (i, j = 1, 2, 3, \cdots),$$

which follows from (2) and (8), and the use of the inequality (11) in proving the uniform convergence of the series that corresponds to the series in the second term of (40).

A corollary to the present theorem that bears the same relation to it that the corollary to Theorem II does to that theorem may be established without difficulty.

§ 3. *The Summability of the Double Fourier's Series at Points of Continuity of the Function Developed.**

We will begin this section by stating that wherever we refer to a function as being finite and integrable, the statement is to be taken in the sense that the function has an integral according to the definition of LEBESGUE.

We will now proceed to prove several lemmas that are needed in the proof of our principal theorem.

LEMMA 5. *Let* R *be a region in the* α, β*-plane lying within the square whose*

* The convergence of the double Fourier's series has been studied by KRAUSE, HARDY, VERGERIO, and W. H. YOUNG. Cf. YOUNG's memoir in the Proceedings of the London Mathematical Society, (ser. 2, vol. 11 (1912), p. 133), where reference to the other writers is made. In the paper just cited, YOUNG has also considered the summability, and has obtained the result stated in Theorem IV of this section. The present paper was completed and sent to editors some time before the appearance of YOUNG's paper, and the results of this section were indicated in an abstract of part of the paper published in the Bulletin of the American Mathematical Society (vol. 18 (1912), p. 223), also before the appearance of YOUNG's paper. The writer has also studied the summability of the double Fourier's series at points of discontinuity of the function developed, as is indicated in the abstract just referred to and in a note published in the Comptes Rendus (vol. 154 (1912), p. 126). The results obtained in this connection will be incorporated in another paper.

sides are $\alpha = \pm(\pi - \rho_1)$, $\beta = \pm(\pi - \rho_1)$ *where* ρ_1 *is a small positive quantity, and such that no point of* R *lies within the circle whose center is at the origin and whose radius is* ρ_2, *where* ρ_2 *is another small positive quantity. Then, if* $\varphi(\alpha, \beta)$ *is a function that is finite and integrable in the region* R, *the limit*

$$\lim_{m,\, n=\infty} \left[\frac{1}{mn} \int\int_R \varphi(\alpha, \beta) \frac{\sin^2 m\alpha}{\sin^2 \alpha} \cdot \frac{\sin^2 n\beta}{\sin^2 \beta} d\alpha\, d\beta \right] \tag{58}$$

will exist and be equal to zero.

Let us represent by M the upper limit of the absolute value of $\varphi(\alpha, \beta)$ in the region R, and by ρ the smaller of the two positive quantities ρ_1 and $\rho_2/\sqrt{2}$. Let ϵ be a given positive quantity, arbitrarily small, and let q be a positive integer such that

$$\frac{4\pi^2 M}{q \sin^2 \rho} < \frac{\epsilon}{2}. \tag{59}$$

We will show that for value of m and n greater than q, the absolute value of the expression in brackets in (58) is less than ϵ. Thus our lemma will be established.

Let us divide the region R into two parts, R_1 and R_2, such that R_1 contains all the points for which $|\alpha| < \rho_2/\sqrt{2}$, and R_2 contains all the points for which $|\alpha| \geqq \rho_2/\sqrt{2}$. It may happen that there are no points such that $|\alpha| < \rho_2/\sqrt{2}$; in that case R_2 will coincide with R. For any point of R we have from the conditions of our lemma.

$$\alpha^2 + \beta^2 \geqq \rho_2^2,$$

and since for the points of R_1

$$\alpha^2 < \frac{\rho_2^2}{2},$$

it follows on subtracting the second inequality from the first that

$$\beta^2 > \frac{\rho_2^2}{2} \geqq \rho^2$$

for all points in R_1. We have then

$$\left| \frac{1}{mn} \int\int_{R_1} \varphi(\alpha, \beta) \frac{\sin^2 m\alpha}{\sin^2 \alpha} \frac{\sin^2 n\beta}{\sin^2 \beta} d\alpha\, d\beta \right|$$

$$\geqq \frac{1}{mn} \int\int_{R_1} |\varphi(\alpha, \beta)| \frac{\sin^2 m\alpha}{\sin^2 \alpha} \frac{\sin^2 n\beta}{\sin^2 \beta} d\alpha\, d\beta \tag{60}$$

$$< \frac{M}{mn \sin^2 \rho} \int\int_{R_1} \frac{\sin^2 m\alpha}{\sin^2 \alpha} d\alpha\, d\beta.$$

But

$$\frac{1}{m}\int\int_{R_1}\frac{\sin^2 m\alpha}{\sin^2\alpha}\,d\alpha\,d\beta < \frac{1}{m}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\sin^2 m\alpha}{\sin^2\alpha}\,d\alpha\,d\beta = 4\pi^2, \tag{61}$$

a result which follows easily from the relation

$$\frac{\pi}{2} = \frac{1}{n}\int_0^{\pi/2}\frac{\sin^2 n\alpha}{\sin^2\alpha}\,d\alpha$$

established in Fejér's paper.* Hence, combining (60), (61), and (59) we obtain

$$\left|\frac{1}{mn}\int\int_{R_1}\varphi(\alpha,\beta)\frac{\sin^2 m\alpha}{\sin^2\alpha}\cdot\frac{\sin^2 n\beta}{\sin^2\beta}\,d\alpha\,d\beta\right| < \frac{4\pi^2 M}{n\sin^2\rho} < \frac{\epsilon}{2} \quad (n \geqq q). \tag{62}$$

If R_1 has an area equal to zero, it is obvious that (62) still holds.

Since for all points of R_2, $|\alpha| \leqq \rho_2/\sqrt{2}$, we may obtain in an entirely similar manner

$$\left|\frac{1}{mn}\int\int_{R_1}\varphi(\alpha,\beta)\frac{\sin^2 m\alpha}{\sin^2\alpha}\cdot\frac{\sin^2 n\beta}{\sin^2\beta}\,d\alpha\,d\beta\right| < \frac{4\pi^2 M}{m\sin^2\rho} < \frac{\epsilon}{2} \quad (m \geqq q). \tag{63}$$

Combining (62) and (63) we have

$$\left|\frac{1}{mn}\int\int_{R}\varphi(\alpha,\beta)\frac{\sin^2 m\alpha}{\sin^2\alpha}\cdot\frac{\sin^2 n\beta}{\sin^2\beta}\,d\alpha\,d\beta\right| < \epsilon \quad (m, n \geqq q),$$

and as pointed out before, our lemma is established.

LEMMA 6. *If h, h_1, k, and k_1 are positive numbers less than π, the limit*

$$\lim_{m,\,n=\infty}\left[\frac{1}{mn\pi^2}\int_{-h_1}^{h}\int_{-k_1}^{k}\frac{\sin^2 m\alpha}{\sin^2\alpha}\cdot\frac{\sin^2 n\beta}{\sin^2\beta}\,d\alpha\,d\beta\right] \tag{64}$$

will exist and be equal to unity.

The expression in brackets in (64) may be written in the form

$$\frac{1}{mn\pi^2}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\frac{\sin^2 m\alpha}{\sin^2\alpha}\cdot\frac{\sin^2 n\beta}{\sin^2\beta}\,d\alpha d\beta \pm \frac{1}{mn}\int\int_R\frac{1}{\pi^2}\cdot\frac{\sin^2 m\alpha}{\sin^2\alpha}\cdot\frac{\sin^2 n\beta}{\sin^2\beta}\,d\alpha d\beta \tag{65}$$

where R is used to represent the region that must be added to or subtracted from the square whose sides are $\alpha = \pm\frac{1}{2}\pi$, $\beta = \pm\frac{1}{2}\pi$, in order to produce the rectangle whose sides are $\alpha = h$, $\alpha = -h_1$, $\beta = k$, $\beta = -k_1$. It follows at once from Lemma 5 that the limit of the second term in (65), as m and n become infinite, is zero, and hence it only remains to determine the limit of the first term.

* L. c., p. 55. Also CARSLAW, *Introduction to the Theory of Fourier's Series and Integrals, etc.*, p. 142. In FEJÉR's original paper there is a misprint in some of the formulæ on the page quoted, π occurring instead of $\frac{1}{2}\pi$.

That term may be written in the form

$$\left\{\frac{1}{m\pi}\int_{-\pi/2}^{\pi/2}\frac{\sin^2 m\alpha}{\sin^2\alpha}\,d\alpha\right\}\left\{\frac{1}{n\pi}\int_{-\pi/2}^{\pi/2}\frac{\sin^2 n\beta}{\sin^2\beta}\,d\beta\right\}.$$

But we know from Fejér's work * that the value of this expression for any positive, integral values of m and n is unity. Hence its limit as m and n become infinite is unity, and since the limit of the second term in (65) is zero, it follows that the limit of (65), as m and n become infinite, is unity.

LEMMA 7. *Let R be a region in the α, β-plane lying within the square whose sides are $\alpha = \pm(\pi - \rho_1)$, $\beta = \pm(\pi - \rho_1)$ where ρ_1 is a small positive quantity, and such that the point $\alpha = 0, \beta = 0$ lies within or on the boundary of R. Then if $\varphi(\alpha, \beta)$ is a function that is finite and integrable in R, the limit*

$$\lim_{m,\,n=\infty}\left[\frac{1}{mn}\int\int_R \varphi(\alpha,\beta)\frac{\sin^2 m\alpha}{\sin^2\alpha}\cdot\frac{\sin^2 n\beta}{\sin^2\beta}\,d\alpha\,d\beta\right] \tag{66}$$

will exist and be equal to zero, provided

$$\lim_{\alpha,\,\beta=0}[\varphi(\alpha,\beta)] = 0. \tag{67}$$

In view of (67) we may choose ρ_2 so small that

$$|\varphi(\alpha,\beta)| < \frac{\epsilon}{2\pi^2} \qquad (\alpha^2+\beta^2 < \rho_2^2), \tag{68}$$

where ϵ is an arbitrarily small, positive quantity. Let us divide R into parts, R_1 and R_2, where R_1 is the circle with center at the origin and radius equal to ρ_2, or such part of that circle as is included in R, and R_2 is the rest of R. It follows at once from Lemma 5 that

$$\lim_{m,\,n=\infty}\left[\frac{1}{mn}\int\int_{R_2}\varphi(\alpha,\beta)\frac{\sin^2 m\alpha}{\sin^2\alpha}\cdot\frac{\sin^2 m\beta}{\sin^2\beta}\,d\alpha\,d\beta\right] = 0,$$

and hence we can choose q so large that

$$\left|\frac{1}{mn}\int\int_{R_2}\varphi(\alpha,\beta)\frac{\sin^2 m\alpha}{\sin^2\alpha}\frac{\sin^2 n\beta}{\sin^2\beta}\,d\alpha\,d\beta\right| < \frac{\epsilon}{2} \quad (m, n \geqq q). \tag{69}$$

Moreover, in view of (68),

* See the previous footnote.

$$\left|\frac{1}{mn}\int\int_{R_1}\varphi(\alpha,\beta)\frac{\sin^2 m\alpha}{\sin^2\alpha}\cdot\frac{\sin^2 n\beta}{\sin^2\beta}\,d\alpha\,d\beta\right|$$

$$\leqq \frac{1}{mn}\int\int_{R_1}|\varphi(\alpha,\beta)|\cdot\frac{\sin^2 m\alpha}{\sin^2\alpha}\cdot\frac{\sin^2 n\beta}{\sin^2\beta}\,d\alpha\,d\beta$$

$$(70)\qquad < \frac{\epsilon}{2\pi^2}\left\{\frac{1}{mn}\int\int_{R_1}\frac{\sin^2 m\alpha}{\sin^2\alpha}\cdot\frac{\sin^2 n\beta}{\sin^2\beta}\,d\alpha\,d\beta\right\}$$

$$< \frac{\epsilon}{2\pi^2}\left\{\frac{1}{mn}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\frac{\sin^2 m\alpha}{\sin^2\alpha}\cdot\frac{\sin^2 n\beta}{\sin^2\beta}\,d\alpha\,d\beta\right\}$$

$$= \frac{\epsilon}{2\pi^2}\left[\frac{1}{m}\int_{-\pi/2}^{\pi/2}\frac{\sin^2 m\alpha}{\sin^2\alpha}\,d\alpha\right]\cdot\left[\frac{1}{n}\int_{-\pi/2}^{\pi/2}\frac{\sin^2 n\beta}{\sin^2\beta}\,d\beta\right] = \frac{\epsilon}{2}.*$$

Combining (69) and (70) we have

$$\left|\frac{1}{mn}\int\int_{R}\varphi(\alpha,\beta)\frac{\sin^2 m\alpha}{\sin^2\alpha}\cdot\frac{\sin^2 n\beta}{\sin^2\beta}\,d\alpha\,d\beta\right| < \epsilon \qquad (m, n \geqq q),$$

from which our lemma follows at once.

Remark.—In the three previous lemmas it was not found necessary to place any restriction on the behavior of the ratio of m and n as these two quantities become infinite. It may be noted further that the conclusions of these lemmas hold equally well if, instead of letting m and n become infinite together, we let m become infinite first, holding n fixed, and then let n become infinite, or vice versa. We have not insisted on this latter point in the proofs of the lemmas, because it is not of much importance in the applications we have in view.

We are now ready to consider the summability of the development of a function of two variables, $f(x, y)$, in a double Fourier's series, that is the summability of the double series

$$(71)\qquad \sum_{m=1,\,n=1}^{\infty,\,\infty}\frac{1}{2^{E(1/m)+E(1/n)}\pi^2}\int_{-\pi}^{+\pi}\int_{-\pi}^{\pi}f(x', y')\,P_{mn}(x, y, x', y')\,dx'\,dy'$$

$$(72)\qquad P_{mn}(x, y, x', y') = \cos[(m-1)(x'-x)]\cos[(n-1)(y'-y)]$$

$E(z)$ being used as in the theory of numbers to represent the largest integer contained in z.

THEOREM IV. *If the function $f(x, y)$ is finite and integrable in the region*

$$(73)\qquad (-\pi \leqq x \leqq \pi,\ -\pi \leqq y \leqq \pi),$$

the development of the function in a double Fourier's series will be summable

* See the previous footnote.

(C1) to the value of the function at every interior point of the region (73) *at which the function is continuous.*

We have for the series (71)*

$$\frac{S^{(1)}_{mn}(x,\,y)}{A^{(1)}_{mn}} = \frac{S^{(1)}_{mn}(x,\,y)}{mn}$$

$$= \frac{1}{4mn\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x',y')\left(\frac{\sin\dfrac{m(x'-x)}{2}}{\sin\dfrac{x'-x}{2}}\right)^2\left(\frac{\sin\dfrac{n(y'-y)}{2}}{\sin\dfrac{y'-y}{2}}\right)^2 dx'dy'.$$

If we set

$$x'-x=2\alpha, \qquad y'-y=2\beta, \tag{74}$$

we obtain

$$\frac{S^{(1)}_{mn}(x,\,y)}{mn} = \frac{1}{mn\pi^2}\int_{-(\pi+x)/2}^{(\pi-x)/2}\int_{-(\pi+y)/2}^{(\pi-y)/2} f(x+2\alpha,\,y+2\beta)\,\frac{\sin^2 m\alpha}{\sin^2\alpha}\cdot\frac{\sin^2 n\beta}{\sin^2\beta}\,d\alpha d\beta. \tag{75}$$

If we can show that the expression on the right-hand side of (75) approaches $f(x,y)$ as a limit as m and n become infinite, at any interior point of the region (73) at which $f(x,y)$ is continuous, our theorem will be established.

Let us put

$$\varphi(\alpha,\,\beta) = f(x+2\alpha,\,y+2\beta) - f(x,\,y). \tag{76}$$

The right-hand side of (75) may then be written in the form

$$\frac{f(x,y)}{nm\pi^2}\int_{-(\pi+x)/2}^{(\pi-x)/2}\int_{-(\pi+y)/2}^{(\pi-y)/2}\frac{\sin^2 m\alpha}{\sin^2\alpha}\cdot\frac{\sin^2 n\beta}{\sin^2\beta}\,d\alpha\,d\beta$$
$$+\frac{1}{mn\pi^2}\int_{-(\pi+x)/2}^{(\pi-x)/2}\int_{-(\pi+y)/2}^{(\pi-y)/2}\varphi(\alpha,\beta)\,\frac{\sin^2 m\alpha}{\sin^2\alpha}\cdot\frac{\sin^2 n\beta}{\sin^2\beta}\,d\alpha\,d\beta. \tag{77}$$

But from its definition $\varphi(\alpha,\beta)$ is finite and integrable in the region of integration of the integrals in (77), and moreover

$$\lim_{\alpha,\,\beta=0}[\varphi(\alpha,\beta)] = 0,$$

provided $f(x,y)$ is continuous at the point (x,y). It therefore follows from Lemma 7 that the second term in (77) approaches zero as a limit as m and n become infinite, provided (x,y) is an interior point of the region (73) and $f(x,y)$ is continuous at that point. Moreover, it follows from Lemma 6 that the first term in (77), and therefore the whole expression (77), approaches as a limit $f(x,y)$ as m and n become infinite. But since (77) is equal to the right-hand side of (75), it follows that the limit of that expression as m and n become infinite is $f(x,y)$. Our theorem is, therefore, proved.

* This follows at once from the corresponding reduction for the ordinary Fourier's series. See FEJÉR, l. c., p. 54. Also CARSLAW, l. c., p. 141.

COROLLARY 1. *If $f(x, y)$ satisfies the conditions of Theorem IV, its Fourier's development will be uniformly summable to $f(x, y)$ throughout any region whose boundary is interior to the boundary of a region of continuity of $f(x, y)$.*

If we make use of the principle that a function of two variables is uniformly continuous in any closed region in which it is continuous, it requires only a slight modification of the reasoning in Lemmas 5 and 7 and Theorem IV to establish this corollary.

Before stating the second corollary to Theorem IV we wish to introduce a definition which will serve to simplify somewhat its statement. *If a double series has each of its rows summable (Ck), and the series formed from the values of the rows is summable (Ck), we will say that the double series is summable (Ck) by rows, and we will ascribe to it the value of the series formed from the values of the rows. We define in a similar manner a double series summable (Ck) by columns, and the value of such a series.*

We are now ready to state the following corollary:

COROLLARY 2. *If $f(x, y)$ satisfies the conditions of Theorem IV, its Fourier's development will be summable $(C1)$ by rows or by columns at every interior point of the region (73) at which $f(x, y)$ is continuous, and to the value of the function.*

The proof of this corollary * follows from the remark at the end of Lemma 7 in the same manner that the proof of Theorem IV follows from Lemmas 5, 6, and 7.

For the applications to mathematical physics we have in view, we need to know something about the behavior of the Fourier's development of $f(x, y)$ throughout the whole region (73) in cases where there are points of discontinuity of the function in the region. The necessary information is embodied in the following theorem:

THEOREM V. *If $f(x, y)$ satisfies the conditions of Theorem IV, we shall have for its Fourier's development*

$$L < \frac{S_{mn}^{(1)}(x, y)}{mn} < M \quad \left(\begin{matrix} m, n = 1, 2, 3, \cdots \\ -\pi \leqq x \leqq \pi, \ -\pi \leqq y \leqq \pi \end{matrix} \right), \tag{78}$$

where M and L are the upper and lower limits of $f(x, y)$ in the region (73).

In view of equation (75), we have

$$\frac{L}{mn\pi^2} \int_{-(\pi+x)/2}^{(\pi-x)/2} \int_{-(\pi+y)/2}^{(\pi-y)/2} \frac{\sin^2 m\alpha}{\sin^2 \alpha} \cdot \frac{\sin^2 n\beta}{\sin^2 \beta} \, d\alpha \, d\beta$$

$$< \frac{S_{mn}^{(1)}(x, y)}{mn} < M \int_{-(\pi+x)/2}^{(\pi-x)/2} \int_{-(\pi+y)/2}^{(\pi-y)/2} \frac{\sin^2 m\alpha}{\sin^2 \alpha} \cdot \frac{\sin^2 n\beta}{\sin^2 \beta} \, d\alpha \, d\beta .$$

* We should note that this corollary is also an almost immediate consequence of FEJÉR'S theorem about the summability of an ordinary Fourier's series, for it follows at once from two successive applications of that theorem. However, we cannot in this way obtain Theorem IV from FEJÉR'S theorem, for a double series summable by rows or columns is not necessarily summable as a double series, a fact that may be inferred at once from a theorem due to PRINGSHEIM (cf. Münchener Berichte (1897), p. 107).

But

$$\int_{-(\pi+x)/2}^{(\pi-x)/2}\int_{-(\pi+y)/2}^{(\pi-y)/2}\frac{\sin^2 m\alpha}{\sin^2\alpha}\cdot\frac{\sin^2 n\beta}{\sin^2\beta}\,d\alpha\,d\beta$$

$$=\left(\int_{-(\pi+x)/2}^{(\pi-x)/2}\frac{\sin^2 m\alpha}{\sin^2\alpha}\,d\alpha\right)\left(\int_{-(\pi+y)/2}^{(\pi-y)/2}\frac{\sin^2 n\beta}{\sin^2\beta}\,d\beta\right)=1,*$$

and therefore our theorem follows at once.

§ 4. *Application.*

We will now illustrate the application of the above results to problems of mathematical physics by considering a problem in the flow of heat. Suppose we wish to determine at any instant the temperature of any point of a thin rectangular plate, whose perimeter is kept at the temperature zero and whose initial temperature we know. Let b be the length and c the breadth of the plate, and $f(x, y)$ be a function of two variables that gives us the initial temperature of the plate at each point when we take the lower left-hand corner of the plate as the origin and the sides of lengths b and c as x and y axes respectively. Then the ordinary, formal method of constructing a solution for problems of this type gives us for the temperature of any point at any instant †

$$(79)\qquad \sum_{m=1,\,n=1}^{\infty,\,\infty} u_{mn}(x, y, t)=\frac{4}{bc}\sum_{m=1,\,n=1}^{\infty,\,\infty} a_{mn}\,e^{-a^2\pi^2(m^2/b^2+n^2/c^2)t}\sin\frac{m\pi x}{b}\sin\frac{n\pi y}{c},$$

where

$$(80)\qquad a_{mn}=\int_0^b\int_0^c f(x', y')\sin\frac{m\pi x'}{b}\sin\frac{n\pi y'}{c}\,dx'\,dy'.$$

In order to prove that the double series (79) is *really* the desired solution, we must show that it converges and defines a continuous function of x, y, and t in the region

$$(81)\qquad 0\leqq x\leqq b,\qquad 0\leqq y\leqq c,\qquad t>0,$$

that it satisfies throughout this region the equation

$$(82)\qquad \frac{\partial u}{\partial t}=a^2\left(\frac{\partial^2 u}{dx^2}+\frac{\partial^2 u}{\partial y^2}\right),$$

that it approaches $f(x_1, y_1)$ as t approaches zero through positive values and

* See FEJÉR, l. c., p. 60.

† Cf. BYERLY, *An Elementary Treatise on Fourier's Series*, etc., p. 142, ex. 4. An analogous problem in three dimensions that can be treated rigorously by means of the generalizations referred to in the Introduction, in a manner precisely similar to the present treatment, may be found in CARSLAW, *Fourier's Series and Integrals*, p. 307.

x and y approach x_1 and y_1, provided (x_1, y_1) is a point in the interior of a region throughout which $f(x, y)$ is continuous, and that it remains finite when t approaches zero through positive values and x and y approach the coördinates of any point of the region

$$0 \leqq x \leqq b, \qquad 0 \leqq y \leqq c. \tag{83}$$

That the series (79) is convergent, and in fact absolutely convergent in the region (81), provided $f(x, y)$ is finite and integrable in the region (83), is easily shown. For, on account of the presence of the convergence factor

$$e^{-a^2\pi^2(m^2/b^2+n^2/c^2)t}, \tag{84}$$

we have for the general term, $u_{mn}(x, y, t)$, of the series

$$|u_{mn}(x, y, t)| < \frac{K}{m^2 n^2} \quad (0 \leqq x \leqq b,\ 0 \leqq y \leqq c,\ 0 < t_0 \leqq t), \tag{85}$$

where K is a positive constant and t_0 a positive constant as small as we please, provided $f(x, y)$ is finite and integrable in (83). Since the right-hand side of (85) is the general term of a convergent double series of positive terms, it follows that (79) converges absolutely in the region

$$0 \leqq x \leqq b, \qquad 0 \leqq y \leqq c, \qquad 0 < t_0 \leqq t, \tag{86}$$

and therefore, since t_0 is arbitrary, in the region (81). That the function it defines there is continuous can also be shown by means of the inequality (85) for it follows from this inequality and the test for uniform convergence stated at the end of § 1, that (79) is uniformly convergent in the region (86). Since also its terms are continuous there, it follows that it defines a continuous function in (86), or since t_0 is arbitrary, in (81).

We wish next to show that the function defined by (79) satisfies the equation (82) throughout the region (81). Each term of the series obviously satisfies this equation, and hence if we can show that we have a right to form the several derivatives involved by differentiating the series term by term, it will follow that the function defined by the series satisfies this equation. The proof that we have a right to differentiate the series term by term is similar to the proof of the continuity of the function defined by the series, the principal difference being that the present proof rests on the uniform convergence of the derived series and the theorem that we have a right to differentiate a convergent double series if the resulting series is uniformly convergent in the region under consideration. The details of this proof are easily carried through.

It remains finally to show that when t approaches zero through positive

values, and x and y approach x_1 and y_1, the function defined by the series will approach $f(x_1, y_1)$ when (x_1, y_1) is an interior point of a region of continuity of $f(x, y)$, and will remain finite when this is not the case.

In order to establish these facts we must make use of some of the results we have obtained with regard to convergence factors in double series. It will be observed that the convergence factors of the form (84) are, for any particular values of m and n, functions of a single variable t, whereas the convergence factors we have dealt with previously were functions of two variables.* However, there was nothing in the conditions of the theorems on convergence factors to prevent there being a relation between the two arguments of the convergence factors, and if we set

$$\alpha = \frac{a^2 \pi^2 t}{b^2}, \qquad \beta = \frac{a^2 \pi^2 t}{c^2}, \tag{87}$$

the convergence factors of the form (84) assume the form

$$e^{-(m^2\alpha + n^2\beta)}, \tag{88}$$

a form to which the conditions of Theorem II can be applied at once.

Our results on convergence factors must be used in connection with some of the results we have obtained with regard to the summability of the double Fourier's series, in order to establish the facts in question. It follows from Corollary 1 of Theorem IV by a change of variable, that the series (79) without the convergence factors is uniformly summable $(C1)$ to $f(x, y)$ throughout some small region surrounding any interior point of a region of continuity of $f(x, y)$. Hence if we can show that the convergence factors of the form (88) satisfy the conditions of Theorem II, it will follow from the corollary of that theorem that the function defined by (79) approaches $f(x, y)$ uniformly throughout the small region above referred to, as the α and β defined by (87) approach $+0$, or what is the same thing, as t approaches $+0$. From this we shall be able to conclude at once that the function defined by (79) approaches $f(x_1, y_1)$ as x and y approach x_1 and y_1 and t approaches $+0$, provided (x_1, y_1) is an interior point of a region of continuity of $f(x, y)$.

It only remains then, in order to establish the result just stated, to show that the convergence factors of the form (88) satisfy the conditions of Theorem II. These conditions include the conditions of Lemma 3 and the conditions (e) and (f). It is obvious that conditions (e) and (f) and condition (d) of Lemma 3 are satisfied; it remains to be shown that conditions (a), (b), and (c) of Lemma 3 are satisfied.

* Our object in considering convergence factors that were functions of two variables was, of course, to obtain results of greater generality.

Let us consider first condition (*b*). We have obviously

$$|e^{-(m^2\alpha_0+n^2\beta_0)}| < \frac{K}{m^4 n^4} \qquad (m, n = 1, 2, 3, \cdots),$$

where K is a positive constant, provided α_0 and β_0 are any constants greater than zero. Hence

$$\sum_{j=\mu}^{j=\nu} j\,|e^{-(m^2\alpha_0+j^2\beta_0)}| < \sum_{j=\mu}^{j=\nu} \frac{Kj}{m^4 j^4} < \frac{K}{m^4}\sum_{j=1}^{\infty}\frac{1}{j^3} = \frac{K_1}{m^4},$$

where K_1 is a positive constant, and hence condition (*b*) of Lemma 3 is satisfied by the convergence factors of the form (88). That condition (*c*) is satisfied may be shown in a manner entirely similar, and hence it only remains to be proved that condition (*a*) is satisfied.

We have by direct computation

$$\Delta_2^2 e^{-(m^2\alpha+n^2\beta)} = [e^{-m^2\alpha} - 2e^{-(m+1)^2\alpha} + e^{-(m+2)^2\alpha}]\cdot[e^{-n^2\beta} - 2e^{-(n+1)^2\beta} + e^{-(n+2)^2\beta}]. \tag{89}$$

It follows then that

$$\sum_{\mu_1,\,\nu_1}^{\mu_2,\,\nu_2} ij\,|\Delta_2^2 e^{-(i^2\alpha+j^2\beta)}| = \left[\sum_{i=\mu_1}^{\mu_2} i\,(e^{-i^2\alpha} - 2e^{-(i+1)^2\alpha} + e^{-(i+2)^2\alpha})\right] \cdot \left[\sum_{j=\nu_1}^{\nu_2} j\,(e^{-j^2\beta} - 2e^{-(j+1)^2\beta} + e^{-(j+2)^2\beta})\right],$$

and hence (*a*) will hold for some K provided K can be so chosen that

$$\sum_{i=1}^{\infty} i\,|e^{-i^2u} - 2e^{-(i+1)^2u} + e^{-(i+2)^2u}| < \sqrt{K} \qquad (u > 0). \tag{90}$$

Therefore it only remains to be shown that the inequality (90) holds for a proper choice of K.

We have evidently

$$e^{-i^2u} - 2e^{-(i+1)^2u} + e^{-(i+2)^2u} = e^{-(i+\theta)^2u}\{4(i+\theta)^2u^2 - 2u\} \quad (0 < \theta < 2), \tag{91}$$

whence it follows that the terms in the series

$$\sum_{i=1}^{\infty} i\,(e^{-i^2u} - 2e^{-(i+1)^2u} + e^{-(i+2)^2u}) \tag{92}$$

are negative for $i = 1, 2, \cdots$, so long as $(i+2)$ is less than $(2u)^{-1/2}$, and are positive for all values of i greater than $(2u)^{-1/2}$. Thus the series consists of a group of negative terms followed by two terms of uncertain sign and an infinite set of positive terms. But it follows at once from (91), that a positive constant can be assigned such that each individual term of the series is less in absolute value than this constant. Hence K can be chosen so that the

inequality (90) holds, provided any sequence of positive or of negative terms that we can choose from (92) is less in absolute value than a fixed, positive constant. Any such set of terms

$$p\,(e^{-p^2u} - 2e^{-(p+1)^2u} + e^{-(p+2)^2u}) + \cdots + q\,(e^{-q^2u} - 2e^{-(q+1)^2u} + e^{-(q+2)^2u})$$

has the sum

$$pe^{-p^2u} - (p-1)\,e^{-(p+1)^2u} - (q+1)\,e^{-(q+1)^2u} + qe^{-(q+2)^2u} \tag{93}$$

which differs from

$$p\,(e^{-p^2u} - e^{-(p+1)^2u}) - (q+1)\,(e^{-(q+1)^2u} - e^{-(q+2)^2u}) \tag{94}$$

by at most unity. But

$$\begin{aligned} |\,m\,(e^{-m^2u} - e^{-(m+1)^2u})\,| &= |\,me^{-(m+\theta)^2u}\{-2\,(m+\theta)\,u\}\,| \\ &< 2\,(m+\theta)^2\,ue^{-(m+\theta)^2u} \quad (0<\theta<1), \end{aligned} \tag{95}$$

and since $\rho e^{-\rho}$ is limited for $\rho > 0$, the left-hand member of (95) is limited for all positive, integral values of m and all values of $u > 0$. Hence the same is true of (94) and (93), and therefore any sequence of positive or negative terms chosen from (92) is less in absolute value than a fixed positive constant for all values of $u > 0$. Hence (90) holds, and consequently, as we have pointed out before, condition (a) is satisfied.

We have now shown that the convergence factors of the form (88) satisfy all the conditions of Theorem II. It therefore follows, as we have pointed out before, that the function defined by (79) approaches $f(x_1, y_1)$ as x and y approach x_1 and y_1 and t approaches $+0$, provided (x_1, y_1) is an interior point of a region of continuity of $f(x, y)$. It remains to be shown that the function defined by (79) remains finite as t approaches $+0$ and x and y approach the coördinates of any point in the region (83) in order to complete the proof that this function furnishes the desired solution of our physical problem.

We have seen that the convergence factors of the form (88) satisfy all the conditions of Lemma 3. It follows then from that lemma that for all positive values of α and β, that is to say for all values of $t > 0$, the series

$$\sum_{i=1,\,j=1}^{\infty,\,\infty} S_{ij}^{(1)}(x, y)\,\Delta_2^2\,e^{-(i^2\alpha+j^2\beta)} \tag{96}$$

converges to the same value as the series (79). But, since $f(x, y)$ is finite and integrable in the region (83), it follows from Theorem V that

$$|\,S_{ij}^{(1)}(x, y)\,| < Cij \qquad \begin{pmatrix} i, j = 1, 2, 3, \cdots \\ 0 \leqq x \leqq b,\ 0 \leqq y \leqq c \end{pmatrix},$$

where C is a positive constant. Combining this result with the fact that condition (a) of Lemma 3 is satisfied by the convergence factors in (79), we have

$$\left| \sum_{i=1,j=1}^{\infty,\infty} S_{ij}^{(1)}(x, y)\, \Delta_2^2 e^{-(i^2\alpha+j^2\beta)} \right| \leqq \sum_{i=1,j=1}^{\infty,\infty} |S_{ij}^{(1)}(x, y)| \cdot |\Delta_2^2 e^{-(i^2\alpha+j^2\beta)}|$$

$$< \sum_{i=1,j=1}^{\infty,\infty} Cij\, |\Delta_2^2 e^{-(i^2\alpha+j^2\beta)}| < CK \quad \left(\substack{\alpha,\beta>0 \\ 0\leqq x\leqq b,\ 0\leqq y\leqq c}\right),$$

and hence it follows that the series (96) remains finite for all positive values of α and β and all values of x and y in the region (83). But since the series (79) converges to the same value as the series (96) for all values of $t>0$, it follows that the function defined by the series (79) remains finite as t approaches $+0$ and x and y approach the coördinates of any point in the region (83). Hence the function defined by (79) furnishes the desired solution of our physical problem.

PARIS,
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ALGEBRAIC SURFACES INVARIANT UNDER AN INFINITE DISCONTINUOUS GROUP OF BIRATIONAL TRANSFORMATIONS:

(SECOND PAPER*)

BY

VIRGIL SNYDER

In a recent paper Dr. ROSENBLATT gives two interesting examples of algebraic surfaces which are invariant under an infinite discontinuous group of birational transformations, and at the same time are not envelopes of quadric surfaces.†

In an earlier paper I mentioned ‡ that all surfaces belonging to such groups which have thus far been noticed were examples of surfaces defining an ordinary elliptic (2, 2) correspondence. Practically all the memoirs bearing on the problem are mentioned in the articles just named. The surfaces discussed by DR. ROSENBLATT have a pencil of elliptic curves, and the transformations are expressed by a linear transformation of the parameters u, v of elliptic functions, in terms of which the coördinates of a point on either surface can be rationally expressed. The treatment is transcendental, the transformation is defined only for points on the surface, and no explanation of the geometric meaning of the transformation is given.

It is a curious fact that these transformations are birational for all space, that they have a simple geometric interpretation in terms of the (2, 2) correspondence, and that surfaces of any order or of as high a geometric genus as may be desired can be constructed which are invariant under this group, or a larger group, under which that discussed by Dr. Rosenblatt is a subgroup of infinite index.

Consider the surface whose equation is of the form

$$\sum_{k=0}^{n} F^{n-k} x^k \varphi_{2k}(z, t) = 0,$$

* Presented to the Society January 1, 1913.

† A. ROSENBLATT: *Algebraische Flächen mit diskontinuierlich unendlich vielen birationellen Transformationen in sich*, Rendiconti . . . di Palermo, vol. 33 (1912), pp. 212–216.

‡ *Infinite discontinuous groups of birational transformations which leave certain surfaces invariant*, these Transactions, vol. 11 (1910), pp. 15–24.

wherein

$$F \equiv x^3 - ty\,(y - x)$$

and φ_{2k} is any binary form of order $2k$. The three lines

$$x = 0,\ t = 0;\quad x = 0,\ y = 0;\quad x = 0,\ y - z = 0$$

are each n-fold upon the surface, their common point $(0, 0, 0, 1)$ being a uniplanar point of order n. The line $z = 0, t = 0$ does not lie on the surface; any plane $t = mz$ through this line will cut the surface in n cubic curves belonging to a pencil, the equation of each being of the form

$$x^3 - mzy^2 + mz^2\,y + l_i\,xz^2 = 0,$$

in which l_i is a function of m and of the coefficients in the various binary forms φ. The basis points of the pencil to which these curves belong are all on the line

$$x = 0,\quad t = mz,$$

being at the points at which the plane cuts the multiple lines of the surface.

At $S \equiv (0, 0, 1, m)$, and at $T \equiv (0, 1, 1, m)$ the curves have simple intersection, while at $U \equiv (0, 1, 0, 0)$ each component curve has a point of inflexion with a common inflexional tangent, and all have seven-point contact with each other.

Any point $P \equiv (x_1, y_1, z_1, t_1)$ on the surface will determine a plane of the pencil $t = mz$, and hence a value of m. In this plane there is one and only one cubic curve lying on the surface and passing through P. If the operation of projecting P upon this curve from S is denoted by S, then $S^2 = 1$, and the operation is birational for every point on the surface, since two curves in a plane $t = mz$ do not have any point in common except the basis points S, T, U. We may now define the operations of projecting each component cubic on itself from T and U by T, U respectively, so that $T^2 = 1, U^2 = 1$.

Any pair of involutions, as S, T, will now generate a group of infinite order and the larger group generated by S, T, U will have the former as a sub-group of infinite index.

The equations of these generators are found by the methods of analytic geometry to be

$$S:\quad \begin{aligned} \rho x' &= x\,(tz^2 - x^3),\\ \rho y' &= y\,(ty^2 - x^3),\\ \rho z' &= x^3\,z,\\ \rho t' &= x^3\,t. \end{aligned}$$

$$T: \begin{cases} \rho x' = x\,[t\,(y-z)^2 - x^3\,], \\ \rho y' = t\,(y-z)^3 + 2x^3 z - x^3 y, \\ \rho z' = x^3 z, \\ \rho t' = x^3 t. \end{cases}$$

$$U: \begin{cases} \rho x' = x, \\ \rho y' = z - y, \\ \rho z' = z, \\ \rho t' = t. \end{cases}$$

$$UTU = S.$$

In the preceding illustration, the order of the surface is $3n$ and the line $z = 0$, $t = 0$ does not lie on the surface. If F' is now defined as the product of F by any binary form in z, t, of order p then the equation

$$\sum_{k=0}^{n} F'^{n-k}\, x^k\, \varphi_{2k+\rho}\,(z,\ t) = 0$$

will define a surface of order $3n + p$, still invariant under the same transformations, and having $z = 0$, $t = 0$ as a singular line of multiplicity $n + p$.

In both cases the cubic curves are not uni-modular, but a plane of the pencil can always be found such that one component curve lying in it has a modulus which may be assigned at will.

Moreover, the form of the surface can be varied still further by replacing two of the straight line loci of basis points of the cubic curves by rational curves of order s, having a $(s-1)$-fold point on the axis of the pencil of planes, which may also be a multiple line on the surface. Thus the equation may be written in the form

$$\sum_{k=0}^{n} H^{n-k}\, x^k\, f_{(2s-1)k}\,(z,\ t) = 0,$$

wherein

$$H \equiv x^3\, t^{2s-3} - xzt^{2s-2} - y^2\, t^{2s-2} + z^{2s},$$

and f_i is any binary form.

As in the preceding case, any plane $t = mz$ will cut from the surface a series of cubic curves all belonging to a pencil, the basis points being at the intersections of the line $x = 0$, $t = mz$ with the line $x = 0$, $z = 0$ and with the rational curves

$$yt^{s-1} + z^s = 0, \quad x = 0; \qquad yt^{s-1} - z^s = 0, \quad x = 0.$$

These points are respectively

$$N \equiv (0, 1, 0, 0), \qquad L \equiv (0, 1, m^{s-1}, m^s),$$
$$M \equiv (0, 1, -m^{s-1}, -m^s).$$

Using these same letters to denote the projections of the cubic curves upon themselves from the three collinear points, we obtain the following generating transformations.

$$L: \begin{cases} \rho x' = x t^{s-1} (z^s - t^{s-1} y)^2 - x^4 t^{3s-4}, \\ \rho y' = 2x^3 z^s t^{2s-3} - (z^s - t^{s-1} y)^3 - x^3 y t^{3s-4}, \\ \rho z' = x^3 z t^{3s-4}, \\ \rho t' = x^3 t^{3s-3}. \end{cases}$$

$$M: \begin{cases} \rho x' = x t^{s-1} (z^s + t^{s-1} y)^2 - x^4 t^{3s-4}, \\ \rho y' = -2x^3 z^s t^{2s-3} + (z^s + t^{s-1} y)^3 - x^3 y t^{3s-4}, \\ \rho z' = x^3 z t^{3s-4}, \\ \rho t' = x^3 t^{3s-3}. \end{cases}$$

$$N: \begin{cases} \rho x' = x, \\ \rho y' = -y, \\ \rho z' = z, \\ \rho t' = t. \end{cases}$$

$$NLN = M.$$

CORNELL UNIVERSITY,
June, 1912.

CONCERNING VAN VLECK'S NON-MEASURABLE SET*

BY

N. J. LENNES

The construction of a point-set which is not measurable in the sense of LEBESGUE has been described by VAN VLECK.† In view of the great interest attaching to the question of the existence and structure of such sets, it is thought desirable to formulate the construction with explicit reference to a definite list of postulates. The general procedure will be that of Van Vleck; with minor departures however, to which attention will be called by notes as they occur. The postulates employed are selected from those given by ZERMELO.‡

Postulate I. *If $E(x)$ is a significant proposition for every element x of a set M, then there is a set M_1 which has as elements all those elements x of M for which $E(x)$ is true and no other element.*

Postulate II. *For every set T there is a set UT which has as elements all subsets of T and no other element.*

Postulate III. *For every set T, whose elements are sets, there is a set ST which has as elements all the elements of the elements of T and no other element.*

Postulate IV. *If T is any set whose elements are all sets, each different from the zero set, and such that no two have an element in common, then the set ST contains a subset which contains one and only one element of each element of T (Prinzip der Auswahl).*

Definition. A set of points T is *homogeneous* on an interval as 01 if there exists a set R, everywhere dense on 01, such that if a and b are points of R

* Presented to the Society April 27, 1912.

† E. B. VAN VLECK: *Non-measurable sets of points.* These Transactions, vol. 9 (1908), p. 237.

‡ E. ZERMELO: *Untersuchungen über die Grundlagen der Mengenlehre,* Mathematische Annalen, vol. 65 (1908), p. 261.

then the subset of T which lies on ab is similar to the set T on 01. We also say that the set T is homogeneous with respect to the set R.*

The exterior Lebesgue measure $Me(T)$ of a set T is the greatest lower bound of the sum of the lengths of the segments of a non-overlapping set of segments covering the set T. If the set T is measurable the measure $M(T)$ of the set is equal to its exterior measure. Let C be the complementary set of T on 01. Then if T is measurable, C is measurable and we have $M(T) + M(C) = 1$.

The theorem of Van Vleck. *If a set T is measurable and homogeneous in the interval* 01 *then its measure is either zero or one.*

Proof. Suppose that $M(T) = 1 - \Sigma$ where Σ is a definite number such that $0 < \Sigma < 1$. Let T be homogeneous with respect to a set R. Then there exists a set of non-overlapping segments $[\sigma]$ covering T such that all end-points of segments of $[\sigma]$ are points of R, and further such that $M[\sigma] = 1 - \Sigma + \epsilon$, where ϵ is an arbitrary positive number such that $1 - \Sigma + \epsilon \leqq 1$. Let ab be any segment of $[\sigma]$. Then on ab there is a set of segments $[\sigma]_{ab}$ similar to $[\sigma]$ and covering all points of T on ab. Consequently $1/(1 - \Sigma + \epsilon) = ab/M[\sigma]_{ab}$. A similar set of segments exists on each segment of $[\sigma]$. Denote the set of all these segments by $[\sigma]_l$. Then T is covered by a set of non-overlapping segments $[\sigma]_l$ such that

$$\frac{1}{1 - \Sigma + \epsilon} = \frac{1 - \Sigma + \epsilon}{M[\sigma]_l},$$

or

$$M[\sigma]_l = (1 - \Sigma + \epsilon)^2,$$

Since ϵ is arbitrary, it may be so chosen that $(1 - \Sigma + \epsilon)^2 < 1 - \Sigma$, contrary to the assumption that $M(T) = 1 - \Sigma$.

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* VAN VLECK's definition of homogeneity is as follows: If in any subinterval whatsoever ab of the interval 01 there can be found an interval as nearly equal in length to ab as we please such that the subset of T included in it shall be similar to the set T on 01 then T is homogeneous (loc. cit., p. 238; see also certain modifications). A set which is homogeneous according to the definition of this paper is obviously homogeneous according to Van Vleck's definition. It would seem however that the converse need not be true, though I have not found an example to show this. It should be observed however that the degree of generality of these sets is of no moment for our present purpose. Here it is sufficient so to define homogeneity that the theorem of Van Vleck shall hold of all homogeneous sets, and then to construct two such sets with a peculiar property as regards measures.

We shall now reject, as in Van Vleck's construction, the rational points of the interval 01, and shall separate the remaining points (whose measure is 1) into two homogeneous sets T and C which are superposable and have therefore the same measure either 0 or 1. Hence $M(T) + M(C) = 0$ or 2, and as this is absurd the sets are non-measurable.*

Denote by Q the set of all irrational points of 01. Then for x_1, any point of Q, the sets $[x_1 r_1 + s_1]$ and $[1 - (x_1 r_2 + s_2)]$, (r_1, r_2, s_1, s_2, having as their range all rational numbers with the restriction that $r_1 + r_2 \neq 0$, $s_1 + s_2 \neq 1$), have no element in common. We say that a set of points has the property J if it has as elements all points on 01 of a pair of sets of the type $[x_1 r + s]$, $[1 - (x_1 r + s)]$ and no other element. The point x_1 may be regarded as a distinctive element of a set of having property J. Clearly if any other point of the set is taken as the distinctive element the same set will be generated. If two sets having property J have an element in common, they are identical. The sets $[x_1 r + s]$ and $[1 - (x_1 r + s)]$ are superposable.

By Postulate II there is a set UQ which has as elements all subsets of Q and no other element. For every element of UQ it is a significant proposition whether or not it has the property J. Hence by Postulate I there is a definite subset M of UQ which has as elements all those subsets of Q which have property J, and no other element. Each element of M is a set different from the zero set, and no two of these sets have an element in common. Hence by Postulate IV there is a set N which contains as elements one and only one element of each set of M. But each element x of N may now be regarded as the distinctive element of a set having the property J.

We now consider that subset K of UQ whose elements are of the form $xr + s$, where x is an element of N. By Postulate III there is a set $SK = T$ whose elements are the elements of the elements of K. The remaining irrational points of 01, $Q - T$, we call a set C. The sets T and C are superposable, since one set is a reflection of the other with respect to the point $1/2$. Further, T is homogeneous on 01 with respect to the set of all rational points on this

* VAN VLECK effects the required separation by describing a set of pairs of sets of the type

$$\left[\frac{x}{2^{\pm p}} \pm \frac{m}{2^n}\right], \qquad \left[1 - \frac{x}{2^{\pm p}} \pm \frac{m}{2^n}\right]$$

and making an *arbitrary choice* of one set of each of these pairs to be in T and the other in C. These sets T and C are superposable and homogeneous. We shall effect the separation by citing a postulate of ZERMELO, thus basing it upon an explicitly recognized fundamental proposition.

interval, since the points of T on 01 may be projected into the points of T on any interval ab whose end-points are rational points.* This completes the proof outlined above.

COLUMBIA UNIVERSITY,
April 22, 1912.

* I am indebted to a graduate student, Mr. G. M. Greene, for the details of this argument.

SOME ASYMPTOTIC EXPRESSIONS IN THE THEORY OF NUMBERS*

BY

T. H. GRONWALL

While formerly the research of asymptotic expressions in the theory of numbers was largely confined to the approximate determination of the summatory function (or the mean value) of a given arithmetic function, recent progress in the theory of prime numbers has opened a new field for asymptotic investigations, viz., the research of upper and lower limits of an arithmetic function for large values of its argument.

The first result in this line was obtained by LANDAU:†

$$\liminf_{x=\infty} \frac{\varphi(x)}{\dfrac{x}{\log\log x}} = e^{-C}, \qquad \limsup_{x=\infty} \frac{\varphi(x)}{x} = 1,$$

where $\varphi(x)$ is the number of relative primes to x which are $< x$, and C the Eulerian constant. For the number $T(x)$ of divisors of x, WIGERT‡ has further shown that

$$\liminf_{x=\infty} T(x) = 2, \qquad \limsup_{x=\infty} \frac{\log T(x)}{\dfrac{\log x}{\log\log x}} = \log 2.$$

In the present paper, I propose to give a similar investigation of the function

$$s_\alpha(x) = \sum_{d \mid x} d^\alpha, \tag{1}$$

the sum extending over all divisors d of the integer x. For $\alpha = 0$, $s_\alpha(x) = T(x)$, and as we have

$$s_{-\alpha}(x) = \sum_{d \mid x} d^{-\alpha} = \sum_{dd'=x} d^{-\alpha} = \sum_{dd'=x} d'^{-\alpha} = \sum_{dd'=x} \left(\frac{x}{d}\right)^{-\alpha} = x^{-\alpha} \sum_{dd'=x} d^\alpha$$

or

$$s_{-\alpha}(x) = x^{-\alpha} s_\alpha(x), \tag{2}$$

it is obviously sufficient to consider the case $\alpha > 0$. As 1 and x are divisors

* Presented to the Society (Chicago) April 5, 1912.

† E. LANDAU, *Handbuch der Lehre von der Verteilung der Primzahlen*, Leipzig, 1909, pp. 217–219. In the following, this work will be briefly quoted as "*Handbuch*."

‡ *Handbuch*, pp. 219–222.

of x, we have

$$s_\alpha(x) \lesseqgtr 1 + x^\alpha,$$

the equality sign prevailing when x is a prime number, and consequently

$$\liminf_{x=\infty} \frac{s_\alpha(x)}{x^\alpha} = 1, \qquad \alpha > 0. \tag{3}$$

To obtain the corresponding superior limit, we express $s_\alpha(x)$ in terms of the prime factors of x. We denote the prime numbers, in their natural order, by $p_1 (= 2), p_2 (= 3), p_3 (= 5), \cdots, p_n, \cdots$, and decompose x into prime factors:

$$x = p_{\lambda_1}^{\nu_1} \cdot p_{\lambda_2}^{\nu_2} \cdot \cdots \cdot p_{\lambda_n}^{\nu_n}, \tag{4}$$

where

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n \qquad \text{and} \qquad \nu_1 > 0, \quad \nu_2 > 0, \quad \cdots, \quad \nu_n > 0.$$

We then obviously have

$$s_\alpha(x) = \sum_{\mu_1=0}^{\nu_1} \cdots \sum_{\mu_n=0}^{\nu_n} p_{\lambda_1}^{\alpha\mu_1} \cdots p_{\lambda_n}^{\alpha\mu_n} = \prod_{k=1}^{n} \left(\sum_{\mu_k=0}^{\nu_k} p_{\lambda_k}^{\alpha\mu_k} \right),$$

whence the formula*

$$s_\alpha(x) = \prod_{k=1}^{n} \frac{p_{\lambda_k}^{\alpha(\nu_k+1)} - 1}{p_{\lambda_k}^{\alpha} - 1} = x^\alpha \prod_{k=1}^{n} \frac{1 - \dfrac{1}{p_{\lambda_k}^{\alpha(\nu_k+1)}}}{1 - \dfrac{1}{p_{\lambda_k}^{\alpha}}}. \tag{5}$$

We now distinguish three cases: $\alpha > 1$, $\alpha = 1$, and $0 < \alpha < 1$.

First case, $\alpha > 1$.—This case may be treated in a quite elementary way. From (5) it follows that

$$s_\alpha(x) < x^\alpha \prod_{k=1}^{n} \frac{1}{1 - \dfrac{1}{p_{\lambda_k}^{\alpha}}} < x^\alpha \prod_{p} \frac{1}{1 - \dfrac{1}{p^\alpha}},$$

where the second product extends over all prime numbers,† and the well-known relation‡

$$\prod_{p} \frac{1}{1 - \dfrac{1}{p^\alpha}} = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} = \zeta(\alpha)$$

* This formula may be found in any elementary text book on the theory of numbers.

† In fact, the second product arises from the first by multiplication by all factors $\dfrac{1}{1 - \dfrac{1}{p^\alpha}}$ where p does not divide x, and each of these factors is obviously > 1.

‡ This formula may be found in any elementary text book on the theory of numbers.

for $\alpha > 1$, where $\zeta(\alpha)$ is the Riemann Zeta function, gives

$$s_\alpha(x) < x^\alpha \zeta(\alpha), \tag{6}$$

whence

$$\limsup_{x=\infty} \frac{s_\alpha(x)}{x^\alpha} \leqq \zeta(\alpha). \tag{7}$$

In order to show that the equality sign prevails in (7), the obvious way is to construct a special infinite sequence of integers $x_1, x_2, \cdots x_n, \cdots$ such that

$$\lim_{n=\infty} \frac{s_\alpha(x_n)}{x_n^\alpha} = \zeta(\alpha). \tag{8}$$

To this purpose, make

$$x_{n,\nu} = (p_1 p_2 \cdots p_n)^{\nu-1} \qquad \begin{pmatrix} n = 1, 2, 3, \cdots \\ \nu = 2, 3, 4, \cdots \end{pmatrix};$$

we then obtain from (5)

$$s_\alpha(x_{n,\nu}) = x_{n,\nu}^\alpha \prod_{k=1}^{n} \frac{1 - \frac{1}{p_k^{\nu\alpha}}}{1 - \frac{1}{p_k^\alpha}}. \tag{9}$$

Now the infinite product

$$\prod_p \frac{1 - \frac{1}{p^{\nu\alpha}}}{1 - \frac{1}{p^\alpha}} = \frac{\zeta(\alpha)}{\zeta(\nu\alpha)}$$

is uniformly convergent in respect to ν for $\nu \geqq 2$;* to any given $\epsilon > 0$ we may therefore find an $n(\epsilon)$ independent of ν such that for $n \geqq n(\epsilon)$ and $\nu \geqq 2$

$$\frac{\zeta(\alpha)}{\zeta(\nu\alpha)} + \frac{\epsilon}{2} > \prod_{k=1}^{n} \frac{1 - \frac{1}{p_k^{\nu\alpha}}}{1 - \frac{1}{p_k^\nu}} > \frac{\zeta(\alpha)}{\zeta(\nu\alpha)} - \frac{\epsilon}{2}. \tag{10}$$

Furthermore

$$\lim_{\nu=\infty} \zeta(\nu\alpha) = \lim_{\nu=\infty} \sum_{n=1}^{\infty} \frac{1}{n^{\nu\alpha}} = 1;$$

* For the truth of this statement, it is necessary and sufficient that

$$\sum_p \frac{1}{p^{\nu\alpha}}$$

be uniformly convergent in respect to ν for $\nu \geqq 2$. This is immediately seen to be the case, as for $\nu \geqq 2$, $\alpha > 1$

$$\frac{1}{p^{\nu\alpha}} < \frac{1}{p^2}$$

and $\sum_p \frac{1}{p^2}$ is convergent, each term being also a term in the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

we may therefore determine a $\nu(\epsilon)$ such that for $\nu \gtreqless \nu(\epsilon)$

$$\zeta(\alpha)+\frac{\epsilon}{2}>\frac{\zeta(\alpha)}{\zeta(\nu\alpha)}>\zeta(\alpha)-\frac{\epsilon}{2}. \tag{11}$$

From (9), (10) and (11) it then follows that

$$x_{n,\nu}^{\alpha}(\zeta(\alpha)+\epsilon)>s_{\alpha}(x_{n,\nu})>x_{n,\nu}^{\alpha}(\zeta(\alpha)-\epsilon)$$

for $n \gtreqless n(\epsilon)$ and $\nu \gtreqless \nu(\epsilon)$. Now if we make $\nu = n+1$ and

$$x_n = x_{n,\,n+1} = (p_1 p_2 \cdots p_n)^n$$

it follows that for $n \gtreqless$ the larger of the numbers $n(\epsilon)$ and $\nu(\epsilon)-1$,

$$\zeta(\alpha)+\epsilon>\frac{s_{\alpha}(x_n)}{x_n^{\alpha}}>\zeta(\alpha)-\epsilon,$$

that is, (8) is satisfied, and the combination of (7) and (8) then gives the desired expression

$$\limsup_{x=\infty}\frac{s_{\alpha}(x)}{x^{\alpha}}=\zeta(\alpha), \tag{12}$$

for $\alpha > 1$.

Second case, $\alpha = 1$.—This case (as well as the third one) is not accessible by the elementary method of the first case, but requires the use of some of the simplest problems in the analytical theory of prime numbers. The two principal arithmetic functions used in this theory are $\pi(x)$, denoting the number of primes $\leqq x$, and $\vartheta(x)$, denoting the sum of the logarithms of these primes:

$$\pi(x)=\sum_{p\leqq x}1, \qquad \vartheta(x)=\sum_{p\leqq x}\log p. \tag{13}$$

The main object of the theory in question is the derivation of asymptotic formulæ for $\pi(x)$ and $\vartheta(x)$ when x is large, the simplest results in this direction being

$$\lim_{x=\infty}\frac{\pi(x)\log x}{x}=1, \tag{14}$$

$$\lim_{x=\infty}\frac{\vartheta(x)}{x}=1.* \tag{15}$$

Of the more accurate asymptotic expressions, we shall here need only the following one†

$$\pi(x)=\frac{x}{\log x}+O\left(\frac{x}{\log^2 x}\right).‡ \tag{16}$$

* *Handbuch*, p. 193.

† *Handbuch*, p. 196.

‡ The symbols O and o, which are very useful in asymptotic calculations, are defined in the following manner (*Handbuch*, p. 59–62):

Returning to our problem, we define an integer n_x by the conditions

$$\prod_{k=1}^{n_x} p_k \leqq x < \prod_{k=1}^{n_x+1} p_k; \tag{17}$$

then by (13)

$$\vartheta(p_{n_x}) \leqq \log x < \vartheta(p_{n_x+1}),$$

and as it easily follows from (14) that

$$\lim_{n=\infty} \frac{p_{n+1}}{p_n} = 1, \tag{18}$$

we conclude by the aid of (15) that

$$1 = \lim_{x=\infty} \frac{\vartheta(p_{n_x})}{p_{n_x}} \leqq \lim_{x=\infty} \frac{\log x}{p_{n_x}} \leqq \lim_{x=\infty} \frac{\vartheta(p_{n_x+1})}{p_{n_x}} = \lim_{x=\infty} \frac{p_{n_x+1}}{p_{n_x}} \cdot \lim_{x=\infty} \frac{\vartheta(p_{n_x+1})}{p_{n_x+1}}$$

$$= 1 \cdot 1 = 1,$$

or

$$\lim_{x=\infty} \frac{\log x}{p_{n_x}} = 1, \qquad \lim_{x=\infty} \frac{p_{n_x}}{\log x} = 1,$$

whence finally, by the definition of the symbol o,

$$p_{n_x} = \log x \cdot (1 + o(1)). \tag{19}$$

We furthermore have, C being the Eulerian constant,

$$\prod_{p \leqq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-C}}{\log x}(1 + o(1)),^*$$

The notation $f(x) = O(g(x))$ signifies that $\limsup_{x=\infty} \frac{|f(x)|}{g(x)}$ is finite, so that a positive constant A may be found such that $|f(x)| < Ag(x)$ for all sufficiently large values of x. Examples: $\sqrt{x} = O(x)$, $x + 1 = O(x)$, $1/x^2 = O(1/x^{3/2})$, $\sin x = O(1)$.

The notation $f(x) = o(g(x))$ signifies that $\lim_{x=\infty} \frac{f(x)}{g(x)} = 0$.

Examples: $\log x = o(\sqrt{x})$, $1/x = o(1)$.

These definitions immediately give the following rules for calculation:

I. If $f_1(x) = O(g_1(x))$ and $f_2(x) = O(g_2(x))$, then $f_1(x) + f_2(x) = O(g_1(x) + g_2(x))$ and if $f_1(x) = o(g_1(x))$ and $f_2(x) = o(g_2(x))$, then $f_1(x) + f_2(x) = o(g_1(x) + g_2(x))$.

II. In a sum of several symbols O or o, only the one of the highest order need be retained. Example: If $f(x) = O(x) + O(x \log x) + O(x^2) + O(1) + O(1/x)$, then $f(x) = O(x^2)$.

III. When a is a positive constant, then

$$O(ag(x)) = O(g(x)),$$
$$o(ag(x)) = o(g(x)).$$

IV. From $f_1(x) = O(g_1(x))$ and $f_2(x) = O(g_2(x))$ it follows that $f_1(x) f_2(x) = O(g_1(x) g_2(x))$.

Example: $x = O(x)$, $\sin x = O(1)$, therefore $x \sin x = O(x \cdot 1) = O(x)$.

V. From $f(x) = o(g(x))$ it follows that $f(x) = O(g(x))$.

VI. From $f_1(x) = O(g_1(x))$ and $f_2(x) = o(g_2(x))$ it follows that $f_1(x) f_2(x) = o(g_1(x) g_2(x))$.

* *Handbuch*, p. 139.

whence, substituting p_{n_x} for x and using (19)

$$\prod_{k=1}^{n_x}\left(1-\frac{1}{p_k}\right)=\frac{e^{-C}}{\log\log x+\log(1+o(1))}(1+o(1))$$

$$=\frac{e^{-C}}{\log\log x}(1+o(1)).* \tag{20}$$

For $\alpha=1$, the expression (5) gives

$$s_1(x)=x\prod_{k=1}^{n}\frac{1-\frac{1}{p_{\lambda_k}^{\nu_k+1}}}{1-\frac{1}{p_{\lambda_k}}}<x\prod_{k=1}^{n}\frac{1}{1-\frac{1}{p_{\lambda_k}}}. \tag{21}$$

Now we obviously have $n\leqq n_x$ and $\lambda_k\geqq k$, $p_{\lambda_k}\geqq p_k$, so that

$$\prod_{k=1}^{n}\frac{1}{1-\frac{1}{p_{\lambda_k}}}\leqq\prod_{k=1}^{n}\frac{1}{1-\frac{1}{p_k}}\leqq\prod_{k=1}^{n_x}\frac{1}{1-\frac{1}{p_k}}=e^C\log\log x\cdot(1+o(1))$$

according to (20), and (21) then shows that

$$\limsup_{x=\infty}\frac{s_1(x)}{x\log\log x}\leqq e^C. \tag{22}$$

A special sequence of integers for which this upper limit is effectively reached is obtained by making

$$x_n=(p_1p_2\cdots p_n)^{[\log p_n]}, \tag{23}$$

where $[\log p_n]$ is the greatest integer contained in $\log p_n$. We then find from (4)

$$s_1(x_n)=x_n\prod_{k=1}^{n}\frac{1-\frac{1}{p_k^{[\log p_n]+1}}}{1-\frac{1}{p_k}}>x_n\prod_{p}\left(1-\frac{1}{p^{[\log p_n]+1}}\right)\prod_{k=1}^{n}\frac{1}{1-\frac{1}{p_k}}$$

$$=\frac{x_n}{\zeta([\log p_n]+1)}\prod_{k=1}^{n}\frac{1}{1-\frac{1}{p_k}}=\frac{x_n}{\zeta([\log p_n]+1)}e^C\log p_n(1+o(1)). \tag{24}$$

* According to the definition of $o(1)$, we have

$$\lim_{x=\infty}\log(1+o(1))=\lim_{\epsilon=0}\log(1+\epsilon)=0,\quad\text{or}\quad\log(1+o(1))=o(1)$$

whence

$$\lim_{x=\infty}\frac{\log\log x+\log(1+o(1))}{\log\log x}=1$$

or

$$\log\log x+\log(1+o(1))=\log\log x\cdot(1+o(1)),$$

and

$$\lim_{x=\infty}\frac{1+o(1)}{1+o(1)}=\lim_{\delta,\,\epsilon=0}\frac{1+\delta}{1+\epsilon}=1\quad\text{or}\quad\frac{1+o(1)}{1+o(1)}=1+o(1),$$

whence our formula.

From (23) we obtain

$$\log x_n = [\log p_n]\,\vartheta(p_n) = [\log p_n]\,p_n(1+o(1)) = p_n \log p_n\,(1+o(1)),$$

whence

$$\log\log x_n = \log p_n + \log\log p_n + o(1),$$

$$\lim_{n=\infty} \frac{\log p_n}{\log\log x_n} = 1,$$

and as

$$\lim_{n=\infty} \zeta([\log p_n]+1) = 1, \text{ (24) gives}$$

$$\limsup_{n=\infty} \frac{s_1(x_n)}{x_n \log\log x_n} \leqq e^C.$$

This relation compared with (22) finally shows that

$$\limsup_{x=\infty} \frac{s_1(x)}{x\log\log x} = e^C. \tag{25}$$

Third case, $0<\alpha<1$.—We begin by developing an asymptotic expression for $\log \prod_{p \leqq x} \frac{1}{1-\frac{1}{p^\alpha}}$, where $0<\alpha<1$. As $\pi(n)-\pi(n-1)$ equals 1 or zero according as n is prime or composite, we find by partial summation

$$\begin{aligned}
\log \prod_{p\leqq x} \frac{1}{1-\frac{1}{p^\alpha}} &= -\sum_{p\leqq x} \log\left(1-\frac{1}{p^\alpha}\right) \\
&= -\sum_{n=2}^{x} (\pi(n)-\pi(n-1)) \log\left(1-\frac{1}{n^\alpha}\right) \\
&= \sum_{n=2}^{x} \pi(n)\left(\log\left(1-\frac{1}{(n+1)^\alpha}\right) - \log\left(1-\frac{1}{n^\alpha}\right)\right) \\
&\quad - \pi(x)\log\left(1-\frac{1}{(x+1)^\alpha}\right).
\end{aligned} \tag{26}$$

Now we have, by Taylor's theorem,

$$\log\left(1-\frac{1}{(n+1)^\alpha}\right) - \log\left(1-\frac{1}{n^\alpha}\right) = \frac{\alpha}{n^{1+\alpha}-n} - \frac{1}{2}\cdot\frac{\alpha}{(\xi^{\alpha+1}-\xi)^2}\left((\alpha+1)\xi^\alpha - 1\right)$$

(where $n<\xi<n+1$)

$$\begin{aligned}
&= \frac{\alpha}{n^{1+\alpha}} + O\left(\frac{1}{n^{2+\alpha}}\right) + O\left(\frac{1}{\xi^{2+\alpha}}\right) \\
&= \frac{\alpha}{n^{1+\alpha}} + O\left(\frac{1}{n^{2+\alpha}}\right),
\end{aligned}$$

$$\log\left(1-\frac{1}{(x+1)^\alpha}\right)=-\frac{1}{(x+1)^\alpha}+O\left(\frac{1}{(x+1)^{2\alpha}}\right)$$
$$=-\frac{1}{x^\alpha}+O\left(\frac{1}{x^{1+\alpha}}\right)+O\left(\frac{1}{(x+1)^{2\alpha}}\right)=-\frac{1}{x^\alpha}+O\left(\frac{1}{x^{2\alpha}}\right),$$

and using (16) we obtain

$$\pi(n)\left(\log\left(1-\frac{1}{(n+1)^\alpha}\right)-\log\left(1-\frac{1}{n^\alpha}\right)\right)$$
$$=\left(\frac{n}{\log n}+O\left(\frac{n}{\log^2 n}\right)\right)\left(\frac{\alpha}{n^{1+\alpha}}+O\left(\frac{1}{n^{2+\alpha}}\right)\right)$$
$$=\frac{\alpha}{n^\alpha\log n}+\frac{n}{\log n}O\left(\frac{1}{n^{2+\alpha}}\right)+\frac{\alpha}{n^{1+\alpha}}O\left(\frac{n}{\log^2 n}\right)+O\left(\frac{n}{\log^2 n}\right)O\left(\frac{1}{n^{2+\alpha}}\right)$$
$$=\frac{\alpha}{n^\alpha\log n}+O\left(\frac{1}{n^{1+\alpha}\log n}\right)+O\left(\frac{1}{n^\alpha\log^2 n}\right)+O\left(\frac{1}{n^{1+\alpha}\log^2 n}\right)$$
$$=\frac{\alpha}{n^\alpha\log n}+O\left(\frac{1}{n^\alpha\log^2 n}\right),$$

$$\pi(x)\log\left(1-\frac{1}{(x+1)^\alpha}\right)=\left(\frac{x}{\log x}+O\left(\frac{x}{\log^2 x}\right)\right)\left(-\frac{1}{x^\alpha}+O\left(\frac{1}{x^{2\alpha}}\right)\right)$$
$$=-\frac{x^{1-\alpha}}{\log x}+O\left(\frac{x^{1-2\alpha}}{\log x}\right)+O\left(\frac{x^{1-\alpha}}{\log^2 x}\right)+O\left(\frac{x^{1-2\alpha}}{\log^2 x}\right)$$
$$=-\frac{x^{1-\alpha}}{\log x}+O\left(\frac{x^{1-\alpha}}{\log^2 x}\right).$$

Introducing these approximations in (26), we find

$$\log\prod_{p\leqq x}\frac{1}{1-\frac{1}{p^\alpha}}=\alpha\sum_{n=2}^{x}\frac{1}{n^\alpha\log n}+O\left(\sum_{n=2}^{x}\frac{1}{n^\alpha\log^2 n}\right)+\frac{x^{1-\alpha}}{\log x}+O\left(\frac{x^{1-\alpha}}{\log^2 x}\right).$$

The function $\frac{1}{u^\alpha\log u}$ decreasing monotonously when u increases, we have

$$\int_n^{n+1}\frac{du}{u^\alpha\log u}<\frac{1}{n^\alpha\log n}<\int_{n-1}^{n}\frac{du}{u^\alpha\log u},$$
$$\int_2^{x+1}\frac{du}{u\log u}<\sum_{n=2}^{x}\frac{1}{n^\alpha\log n}<\int_1^x\frac{du}{u^\alpha\log u}$$

or

$$\sum_{n=2}^{x}\frac{1}{n^\alpha\log n}=\int_2^x\frac{du}{u^\alpha\log u}+O(1),$$

and in the same way we obtain an approximation for $\sum_{n=2}^{x} \frac{1}{n^\alpha \log^2 n}$, whence

$$\log \prod_{p \leqq x} \frac{1}{1-\frac{1}{p^\alpha}} = \alpha \int_2^x \frac{du}{u^\alpha \log u} + O(1) + O\left(\int_2^x \frac{du}{u^\alpha \log^2 u}\right) + \frac{x^{1-\alpha}}{\log x} + O\left(\frac{x^{1-\alpha}}{\log^2 x}\right).$$

Integrating by parts, we find

$$\int_2^x \frac{du}{u^\alpha \log u} = \frac{x^{1-\alpha}}{(1-\alpha)\log x} - \frac{2^{1-\alpha}}{(1-\alpha)\log 2} + \frac{1}{1-\alpha}\int_2^x \frac{du}{u^\alpha \log^2 u};$$

we also have

$$\int_2^x \frac{du}{u^\alpha \log^2 u} = \int_2^{\sqrt{x}} + \int_{\sqrt{x}}^x < \frac{1}{\log^2 2}\int_2^{\sqrt{x}} \frac{du}{u^\alpha} + \frac{1}{\log^2 \sqrt{x}}\int_{\sqrt{x}}^x \frac{du}{u^\alpha}$$

$$= O(x^{\frac{1-\alpha}{2}}) + O\left(\frac{x^{1-\alpha}}{\log^2 x}\right) = O\left(\frac{x^{1-\alpha}}{\log^2 x}\right),$$

and introducing in the expression above, we find

$$\log \prod_{p \leqq x} \frac{1}{1-\frac{1}{p^\alpha}} = \frac{\alpha}{1-\alpha}\frac{x^{1-\alpha}}{\log x} + O\left(\frac{x^{1-\alpha}}{\log^2 x}\right) + O(1)$$

$$+ O\left(\frac{x^{1-\alpha}}{\log^2 x}\right) + \frac{x^{1-\alpha}}{\log x} + O\left(\frac{x^{1-\alpha}}{\log^2 x}\right),$$

or finally

$$\log \prod_{p \leqq x} \frac{1}{1-\frac{1}{p^\alpha}} = \frac{1}{1-\alpha}\frac{x^{1-\alpha}}{\log x} + O\left(\frac{x^{1-\alpha}}{\log^2 x}\right) = \frac{1}{1-\alpha}\frac{x^{1-\alpha}}{\log x}(1+o(1)) \tag{27}$$

$$(0 < \alpha < 1).$$

On account of $n \overline{\overline{\leqq}} n_x$ and $p_{\lambda_k} \overline{\overline{\geqq}} p_k$, equation (5) gives

$$s_\alpha(x) < x^\alpha \prod_{k=1}^{n} \frac{1}{1-\frac{1}{p_{\lambda_k}^\alpha}} \overline{\overline{\leqq}} x^\alpha \prod_{k=1}^{n_x} \frac{1}{1-\frac{1}{p_k^\alpha}}. \tag{28}$$

Substituting p_{n_x} for x in (27), we obtain by the aid of (19), manipulating the symbol $o(1)$ in the same way as when deriving (20) from the formula preceding it,

$$\log \prod_{k=1}^{n_x} \frac{1}{1-\frac{1}{p_k^\alpha}} = \frac{1}{1-\alpha}\frac{(\log x)^{1-\alpha}}{\log\log x}(1+o(1)), \tag{29}$$

whence, by (28)

$$\limsup_{x=\infty} \frac{\log \frac{s_\alpha(x)}{x^\alpha}}{\frac{(\log x)^{1-\alpha}}{\log\log x}} \overline{\overline{\leqq}} \frac{1}{1-\alpha}. \tag{30}$$

To obtain a special sequence of integers for which this upper limit is effectively attained, make

$$x_n = p_1 p_2 \cdots p_n .$$

We then obtain from (5)

$$s_\alpha(x_n) = x_n^\alpha \prod_{k=1}^{n} \frac{1 - \frac{1}{p_k^{2\alpha}}}{1 - \frac{1}{p_k^\alpha}}, \tag{31}$$

and we have

$$\log \prod_{k=1}^{n} \left(1 - \frac{1}{p_k^{2\alpha}}\right) = -\frac{1}{1-2\alpha} \frac{(\log x_n)^{1-2\alpha}}{\log\log x_n} (1 + o(1)) \text{ for } 0 < \alpha < \tfrac{1}{2} \text{ (by (29))},$$

$$= -\log\log\log x_n + O(1) \text{ for } \alpha = \tfrac{1}{2} \text{ (by (20))},$$

$$= O(1) \text{ for } \tfrac{1}{2} < \alpha < 1 \text{ (infinite product convergent)},$$

so that in all three cases

$$\log \prod_{k=1}^{n} \left(1 - \frac{1}{p^{2\alpha}}\right) = \frac{(\log x_n)^{1-\alpha}}{\log\log x_n} \cdot o(1).$$

Equations (31) and (29) then give

$$\log \frac{s_\alpha(x_n)}{x_n^\alpha} = \frac{1}{1-\alpha} \frac{(\log x_n)^{1-\alpha}}{\log\log x_n} (1 + o(1))$$

or

$$\lim_{n=\infty} \frac{\log \frac{s_\alpha(x_n)}{x_n^\alpha}}{\frac{(\log x_n)^{1-\alpha}}{\log\log x_n}} = \frac{1}{1-\alpha}. \tag{32}$$

and we finally obtain, by combining (30) and (32)

$$\limsup_{x=\infty} \frac{\log \frac{s_\alpha(x)}{x^\alpha}}{\frac{(\log x)^{1-\alpha}}{\log\log x}} = \frac{1}{1-\alpha} \qquad (0 < \alpha < 1).$$

DETERMINATION OF THE FINITE QUATERNARY LINEAR GROUPS*

BY

HOWARD H. MITCHELL

§ 1. *Introduction.*

A number of writers have concerned themselves with the finite collineation groups in ordinary three-dimensional space, among them JORDAN, KLEIN, MASCHKE, and WITTING.† A complete enumeration of such groups however was first given by BLICHFELDT.‡ His determination of the groups was accomplished largely by means of theorems obtained by him in two previous papers.§ His methods were mainly algebraic, the irreducibility of certain polynomials in given domains playing an important rôle.

About the same time a determination of those groups which contain homologies was given by BAGNERA.‖ The methods employed by him were of geometrical nature, the possible configurations formed by the centers and axial planes of the homologies being considered in detail.

It is the purpose of the present writer to make the determination of all the groups, which do not contain homologies, by geometrical methods, making use of well-known properties of the linear complex, the quadric, etc. The determination applies equally well to the quaternary groups in which the transformations are those of a general modular space,¶ provided the orders of none of the transformations be divisible by the modulus. Bagnera's methods apply also to such groups which contain homologies. A large class of quaternary groups which are determined by the $GF(p)$ and which do contain transformations of period divisible by the modulus have been com-

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† For a résumé of the work done in this field consult WIMAN, *Endliche Gruppen linearer Substitutionen, Encyklopädie der Mathematischen Wissenschaften*, vol. I, pp. 522–554.

‡ Mathematische Annalen, vol. 60 (1905), pp. 204–231.

§ These Transactions, vol. 4 (1903), pp. 387–397, vol. 5 (1904), pp. 310–325.

‖ Rendiconti del Circolo Matematico di Palermo, vol. 19 (1905), pp. 1–56.

¶ Cf. VEBLEN and BUSSEY, *Finite Projective Geometries*, these Transactions, vol. 7 (1906), pp. 241–259.

pletely enumerated by DICKSON.* These papers therefore supplement each other. In the present paper some special groups are noticed incidentally whose orders are divisible by the modulus.

Following BLICHFELDT, we shall apply the term "primitive" to those groups which do not leave invariant a point, a plane, two skew lines, or a tetrahedron. One class of such groups has an invariant quadric surface,† and another class an invariant linear complex. The others are certain groups‡ having an invariant G_{16}, and three simple groups of order 25920, 2520, and 168. The first of these three was found by WITTING.§ It contains homologies of period 3 and hence is in Bagnera's list. The groups which are holoedrically isomorphic with either a symmetric or an alternating group are of orders $\frac{1}{2}7!$, $6!$, $\frac{1}{2}6!$, $5!$ (two types), $\frac{1}{2}5!$ (two types). Those of order $6!$ and $\frac{1}{2}6!$ have an invariant linear complex, and those of order $5!$ and $\frac{1}{2}5!$ either an invariant complex or an invariant quadric. The one of order $5!$ which has an invariant quadric contains homologies of period 2.

§ 2. *Groups which contain homologies.*

Bagnera's results in the case of the groups containing homologies may be summarized in the following theorems. For the proofs and for a detailed discussion of the allied geometrical configurations the reader is referred to his paper.

Theorem 1. *No primitive group contains homologies of period greater than 3.*

Theorem 2. *The only primitive group which contains homologies of period 3 is of order 25920.*

Theorem 3. *No primitive group contains two homologies of period 2, whose product is of period greater than 5.*

Theorem 4. *Groups of order 120 and 288 are the only primitive groups which contain two homologies of period 2, whose product is of period 3, but none whose product is of higher period.*

Theorem 5. *Groups of order 576, 1152, 1920, and 11520 are the only primitive groups which contain two homologies of period 2, whose product is of period 4, but none whose product is of higher period.*

Theorem 6. *A group of order 7200 is the only primitive group which contains two homologies of period 2, whose product is of period 5.*

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*American Journal of Mathematics, vol. 28 (1906), pp. 1–16.

† These groups were first determined by GOURSAT, Annales scientifiques de l'Ecole Normale Supérieure, Ser. 3, vol. 6 (1889), pp. 9–102.

‡ Cf. MASCHKE, Mathematische Annalen, vol. 30 (1887), p. 498.

§ Inaugural-Dissertation, Göttingen (1887), p. 27. Cf. also MASCHKE, Mathematische Annalen, vol. 33 (1889), p. 320.

§ 3. *General properties of groups containing skew perspectivities.*

In the discussion of groups containing skew perspectivities the following facts will be found of use:

Two skew perspectivities must leave invariant in common at least two skew lines, since their product can leave invariant no point unless on a line left invariant by both.

Two skew perspectivities cannot have one axis in common.

If an axis of one skew perspectivity intersects an axis of a second, the other axis of the first must intersect the other axis of the second.

If an axis of one skew perspectivity meets both axes of a second, the other axis of the first must meet both axes of the second.

A linear complex is left invariant by any skew perspectivity whose axes are in the complex.

If a skew perspectivity leaves a linear complex invariant, either its axes are lines of the complex or else it is of period 2 and its axes are axes of a congruence contained by the complex.

Two skew reflections whose axes are axes of congruences in the same linear complex must leave invariant in common all the lines of a regulus. (If the two congruences contained a flat pencil in common the product of the reflections would be an elation in that plane.)

Theorem 7. *In a primitive group a subgroup generated by skew perspectivities of higher period than 2 whose axes lie in a congruence will be contained by a group leaving a single linear complex invariant.*

In a primitive group there will be skew perspectivities conjugate with those whose axes are in the congruence which do not leave the congruence invariant. Such a skew perspectivity must generate with any skew perspectivity whose axes lie in the congruence and whose axes its axes do not intersect either a group leaving invariant all the lines of a regulus or a group leaving a congruence invariant. In the latter case however each skew perspectivity will be contained by more than one group which does leave invariant all the lines of a regulus, and, since the new congruence cannot have more than one regulus in common with the original congruence, there will also be in this case skew perspectivities which do not leave the congruence invariant and which generate with skew perspectivities whose axes do lie in the congruence a group leaving invariant all the lines of a regulus. The axes of such a skew perspectivity would determine a complex together with the congruence.

The truth of these statements follows from the consideration of the various cases which can arise. Two skew perspectivities of period greater than 5, whose axes do not intersect, can generate no group. If they are of period 5, they must generate a group of order 60, which leaves invariant all the lines

of a regulus, or else a group of order $2 \cdot 60 \cdot 60$ in which every skew C_5 lies in five such G_{60}. If they are of period 4, they must generate a group of order 24 leaving invariant all the lines of a regulus. If they are of period 3, they must generate a group of order 12, 60, $2 \cdot 12 \cdot 12$, $2 \cdot 12 \cdot 60$. The group of order 12 leaves invariant all the lines of a regulus; that of order 60 may do so, but if not each C_3 lies in two G_{12} which do; in the groups of order $2 \cdot 12 \cdot 12$ and $2 \cdot 12 \cdot 60$ each C_3 lies in several such G_{12}.*

Theorem 8. *A group leaving invariant a linear complex but not a congruence cannot contain a transformation of even period greater than 4 having a single line of fixed points.*

A power of such a transformation must be a reflection whose axes are axes of a congruence contained by the complex. Any reflection conjugate with this one must leave invariant all the lines of a regulus in common with it and such a regulus will contain two lines left invariant by the transformation having a line of fixed points. Hence, since that transformation is of period not less than 6, any reflection conjugate with it must interchange its axes. But there will be conjugate reflections interchanging those axes which are not themselves commutative.

§ 4. *Groups containing skew perspectivities of period greater than 3.*

Theorem 9. *There is no primitive group which contains skew perspectivities of period greater than 5.*

Two such skew perspectivities which are not commutative can generate no group.

Theorem 10. *No primitive group which does not leave a quadric invariant can contain skew perspectivities of period 5.*

Two skew perspectivities of period 5 which are not commutative must generate a group of order 60 if they leave invariant all the lines of a regulus, a group of order $2 \cdot 5 \cdot 60$ if their axes intersect along one line, or a group of order $2 \cdot 60 \cdot 60$. Each of the last two groups contains transformations of period 10 which leave fixed all the points on one axis of the congruence and are of period 5 on the other. Hence no primitive group can contain them (Theorems 7, 8).

Hence any two skew perspectivities of period 5 which are not commutative must leave invariant all the lines of a regulus. But if a quadric does not remain invariant, not all the skew perspectivities of period 5 which do not lie in a particular G_{60} will be commutative with the G_{60}. Two G_{60} having a skew C_5 in common will leave invariant a congruence and hence must generate a group of order $2 \cdot 60 \cdot 60$.

* For a more detailed discussion of the imprimitive groups the reader may consult BLICHFELDT, these Transactions, vol. 6 (1905), pp. 230–236.

Theorem 11. *A $G_{16\cdot 120}$ containing an invariant G_{16} is the only primitive group containing skew perspectivities of period 4, but no homologies, which does not leave a quadric invariant.*

Two skew perspectivities of period 4 which are not commutative must generate either a group of order $2 \cdot 4 \cdot 12$ (if their axes intersect along one line) or a group of order 24 leaving invariant all the lines of a regulus. A group of the former type contains C_6 which leave fixed all the points on one axis of the congruence and are of period 3 on the opposite axis. Hence it cannot be contained by a primitive group (Theorems 7, 8).

Hence any two skew perspectivities of period 4 which are not commutative must generate a group of order 24 leaving invariant all the lines of a regulus. A G_{24} is generated by

$$[x_1,\ ix_2,\ ix_3,\ x_4] \qquad (i^2 = -1);$$

$$x_1' - x_2' = x_1 - x_2, \qquad x_3' - x_4' = x_3 - x_4,$$

$$x_1' + x_2' = i\,(x_1 + x_2), \qquad x_3' + x_4' = i\,(x_3 + x_4).$$

If no quadric remains invariant, not all the other skew perspectivities of period 4 will be commutative with all of those in this G_{24}. Consider one which is not. Since it must leave invariant all the lines of a regulus in common with a skew C_4 with which it is not commutative, it will leave invariant a congruence in common with the above G_{24}. Since we cannot have two skew C_4 whose axes intersect along a single line, this C_4 must be commutative with a C_4 in the original G_{24}. Hence every C_4 must be commutative with one or all three of the C_4 in that G_{24}. There cannot be three mutually commutative C_4 without involving homologies of period 2. Hence each of the C_4 in the original G_{24} can be commutative with but three C_4 and one of these will be commutative with the G_{24}. There can be but $3 + 3 \cdot 2 + 1 = 10$ C_4 altogether. Two of the others we take to be

$$[x_1,\ ix_2,\ x_3,\ ix_4];$$

$$x_1' - x_4' = x_1 - x_4, \qquad x_1' + x_4' = i\,(x_1 + x_4),$$

$$x_2' - x_3' = x_2 - x_3, \qquad x_2' + x_3' = i\,(x_2 + x_3).$$

A group is generated of order $16 \cdot 120$, leaving invariant the G_{16} which contains the squares of all the ten C_4. The axes of these ten reflections are lines of the invariant complex. The axes of the other five reflections in the G_{16} are axes of congruences which belong to that complex.

§ 5. *Groups containing skew perspectivities of period 3.*

Theorem 12. *The only primitive groups which contain skew perspectivities of period 3 whose axes intersect along a single line, but which contain no skew*

perspectivities of higher period and no homologies, and which do not leave a quadric invariant, are of order 16 · 60 and 16 · 360, each having an invariant G_{16}.

Two skew perspectivities of period 3 which are not commutative must generate a group of one of the following orders: $12, 60, 2 \cdot 3 \cdot 4, 2 \cdot 3 \cdot 12, 2 \cdot 3 \cdot 60, 2 \cdot 12 \cdot 12, 2 \cdot 12 \cdot 60$. We may suppose that the G_{12} is the only group which leaves invariant all the lines of a regulus, since a G_{60} having that property would contain skew perspectivities of period 5. All the above groups except those of order $12, 60, 2 \cdot 3 \cdot 4$ contain C_6 which leave invariant all the points on one axis of the congruence and are of period three on the other axis. Such groups can be contained by no primitive group (Theorems 7, 8).

Consider a group of order $2 \cdot 3 \cdot 4$ generated by two C_3 whose axes intersect along a single line. It contains C_4 which leave fixed all the points on one axis of the congruence and are of period 2 on the other. This group must be contained by a group leaving a linear complex invariant (Theorem 7). In such a group every reflection conjugate with the invariant reflection of the above group must leave invariant all the lines of a regulus in common with it. If the product of two such reflections were a skew C_3, there would be on two of the lines of the invariant regulus a C_4 and a C_3 such that the square of the C_4 would transform the C_3 into its inverse. But there is no such group on a line. Hence any two such reflections must interchange each other's axes. The product of two such reflections moreover must be a reflection whose axes are lines of the complex.

We find readily that there must be five reflections of this sort, which we may take to be $[x_1, x_2, -x_3, -x_4]$, $[x_4, x_3, -x_2, -x_1]$, $[x_4, -x_3, -x_2, x_1]$, $[x_3, x_4, x_1, x_2]$, $[-x_3, x_4, x_1, -x_2]$. The fifth lies in the G_{16} generated by the first four. The whole group leaving invariant the G_{16} and the complex is of order $16 \cdot 120$ and it contains skew perspectivities of period 4. The group in this case must be of order $16 \cdot 60$.

We inquire what groups can contain the G_{960}. The skew perspectivities whose axes lie in one of the five congruences generate a G_{96}. We may show that such a G_{96} must be contained by at least one other G_{960}. For consider a skew C_3 not in the G_{960}. It must generate with a C_3 in the G_{96} whose axes its axes do not intersect either a G_{12} or a G_{60}. In the latter case the C_3 in the G_{96} will be contained by two G_{12} and since the G_{60} can be generated by these two G_{12} not more than one of them can lie in the G_{960}. A G_{12} not in the G_{960} which contains a skew C_3 in the G_{96} will leave a linear complex invariant in common with the G_{96} and hence must generate with the latter group another G_{960}.

We will show that another G_{960} containing the G_{96} must leave invariant the same G_{16}. Hence it will follow that every skew C_3 which generates a G_{12}

with a C_3 in the G_{96} will leave invariant that G_{16}. Since a G_{60} may be generated by the two G_{12} containing a C_3, it must also follow that every C_3 which generates a G_{60} with a C_3 in the G_{96} must leave invariant the same G_{16}. Hence that G_{16} must remain invariant under any larger group.

We proceed to show that any G_{960} containing the G_{96} must leave invariant the same G_{16}. Consider the G_{96} with fixed lines $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$, $x_4 = 0$. It leaves invariant three G_8 which contain no C_4 according as the three pairs of points $(1000)(0100)$, $(1100)(1-100)$, $(1i00)(1-i00)$ are joined by the axes of the reflections to $(0001)(0010)$, $(0011)(00-11)$, $(00i1)(00-i1)$ respectively, or to those three pairs permuted in a cyclic order. If more than one of these G_8 is contained by a G_{16} we will have as the product of reflections which interchange $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$, $x_4 = 0$ and which lie in different G_{16} a transformation which is of period 3 on each of those lines and on the three G_8. This transformation may be of period 6, but its square must be of period 3. This square must be commutative with one of the C_3 in the G_{96}. In the G_9 generated there must be either an homology of period 3 or a skew perspectivity of period 3 whose axes are in the congruence and which is not in the G_{96}. Either case is impossible. Hence the G_{960} which contain the G_{96} must all leave invariant the same G_{16}. The group generated must be of order $16 \cdot 360$.

Theorem 13. *The only primitive groups which contain skew perspectivities of period 3, but none whose axes intersect by pairs, no skew perspectivities of higher period, and no homologies, and which do not leave a quadric invariant are holoedrically isomorphic respectively with the G^5_{360}, the G^6_{720}, and the G^7_{2520}.*

Any two G_{12} having a C_3 in common will leave invariant a congruence and hence must generate a G_{60}. There must be at least three G_{12} containing a C_3 provided a congruence does not remain invariant under the group. Hence a G_{12} must lie in at least two G_{60}. A particular reflection R in the G_{12} will interchange the axes of two C_3 in each G_{60}. Let the two C_3 of this sort in one G_{60} be denoted by A and A', and the two in the other by B and B'.

We will show that A and B cannot generate a G_{60}. For suppose this is the case. R would then leave invariant each of the fixed lines of the G_{60}, for the axes of R, A, and B would lie in a congruence determined by the two axes of R, one axis of A, and one axis of B. Hence either R would lie in the G_{60} or else its product by the C_2 in the G_{60} which interchanges the same two pairs of axes would be a C_2 commutative with the G_{60}.

Consider the first case. R can lie in but one G_{12} leaving invariant all the lines of a regulus. For two such G_{12} would leave invariant a congruence and hence would have to generate a G_{60}. But a C_2 in a G_{60} lies in but one G_{12}. Hence the G_{60} generated by A and B would have to contain the G_{12} which contains R. But the G_{12} and A generate a G_{60} not containing B.

Consider the second case. Since there can be but one G_{12} containing R, the reflection commutative with R, A, and B must leave the G_{12} invariant. If it is commutative with the G_{12} it cannot be commutative with both A and B. It must then be the product of R and a reflection commutative with the G_{12}. It must then be left invariant by the G_4 of the G_{12}. It must then be left invariant by the G_{60} generated by A and this G_4. Consequently it must be left invariant by the G_{12}. But this is impossible unless it is commutative with the G_{12}.

Hence A and B cannot generate a G_{60}. Similarly A and B' cannot generate a G_{60}. It may be shown that A cannot generate a G_{12} with B and another G_{12} with B'. For in this case A, B, and B' would have to generate a G_{60}. But this is impossible since B and B' generate a G_{60} not containing A. Hence A must be commutative with either B or B', say with B'. Since A cannot be commutative with both B and B', it must generate a G_{12} with B.

We take these three C_3 as follows:

$$A\colon [\omega x_1,\ \omega x_2,\ \omega^2 x_3,\ \omega^2 x_4] \qquad (\omega^2+\omega+1=0);$$

$$B\colon \begin{aligned} x_1'-x_4' &= \omega(x_1-x_4), & x_2'-x_3' &= \omega(x_2-x_3),\\ x_1'+2x_4' &= \omega^2(x_1+2x_4), & x_2'+2x_3' &= \omega^2(x_2+2x_3); \end{aligned}$$

$$B'\colon \begin{aligned} x_1'-x_2' &= \omega(x_1-x_2), & x_3' &= \omega^2 x_3,\\ x_1'-ax_2' &= \omega^2(x_1-ax_2), & x_4' &= \omega x_4. \end{aligned}$$

In order that B and B' shall generate a G_{60} we must have $(B'B)^5 = I$. The condition for this is $a = -1$.* The reflection interchanging all three pairs of axes is given by

$$R\colon \begin{aligned} x_1' &= x_3+x_4, & x_3' &= -\tfrac{1}{2}x_1+\tfrac{1}{2}x_2,\\ x_2' &= -x_3+x_4, & x_4' &= -\tfrac{1}{2}x_1-\tfrac{1}{2}x_2. \end{aligned}$$

We then take $E_1 = A$, $E_2 = A^2BA^2$, $E_3 = RB'$,† $E_4 = R$. It may be shown that $I = E_1^3 = E_2^2 = E_3^2 = E_4^2 = (E_1E_2)^3 = (E_1E_3)^2 = (E_1E_4)^2 = (E_2E_3)^3 = (E_2E_4)^2 = (E_3E_4)^3$. These operators therefore generate a group simply isomorphic with the G_{360}^6.‡

Under the G_{360} the G_{12} generated by E_1 and E_2 is left invariant by a G_{24}. Hence if the G_{360} is self-conjugate under a larger group, there must be skew perspectivities commutative with that G_{12}. The only such skew perspectivity which interchanges the axes of B' is the reflection $[-x_2, x_1, x_4, -x_3]$.

* This may be obtained from the fact that the coefficient of ρ^2 in the characteristic equation of $B'B$ must be unity. The coefficient of ρ^3 is -1 for any value of a.

† In a product the order of operating is taken from right to left.

‡ E. H. Moore, Proceedings of the London Mathematical Society, vol. 28 (1897), pp. 357–366. Also Dickson, *Linear Groups*, p. 289.

Hence the G_{360} is self-conjugate under a group of order 720, which must be simply isomorphic with the G^6_{720}.

The G_{360} and the G_{720} leave invariant a linear complex, i. e., that determined by the axes of A and B' and one axis of B.

In a group containing the G_{360} not self-conjugately there must be more C_3. Hence a C_3 must lie in at least four G_{12} and $4 \cdot 3/1 \cdot 2\ G_{60}$. Hence R must interchange the axes of two other reflections, C and C', which together generate a G_{60}. One of them, say C', must be commutative with A. Similarly C' must be commutative with either B or B'. If it is commutative with B' homologies of period 3 will be present. Hence it must be commutative with B. We then have

$$C': \quad \begin{aligned} x_1' - bx_2' &= \omega\ [x_1 - bx_2], \\ (b-1)\,x_1' - (b+1)\,x_2' &= \omega^2[(b-1)\,x_1 - (b+1)\,x_2], \\ x_4' - bx_3' &= \omega\,[x_4 - bx_3], \\ (b-1)\,x_4' - (b+1)\,x_3' &= \omega^2[(b-1)\,x_4 - (b+1)\,x_3]. \end{aligned}$$

In order that B' and C' shall generate a G_{12} we may take $(B'^{-1}C')^2 = I$. The condition for this is $b^2 + b + 2 = 0$. We put $E_5 = E_4 C'$ and find $I = E_5^2 = (E_1 E_5)^2 = (E_2 E_5)^2 = (E_3 E_5)^2 = (E_4 E_5)^3$*. Hence E_5 generates with the G_{360} a group simply isomorphic with the G^7_{2520}.†

If the G_{2520} is contained self-conjugately by a larger group, the G_{360} must be contained self-conjugately by a larger group. The only such group is the G_{720}. Since C' is the only C_3 in the G_{2520} which is commutative with the G_{12} generated by E_1 and E_2, its axes must be interchanged by $[-x_2, x_1, x_4, -x_3]$. But this is not the case.‡ Hence the G_{2520} is self-conjugate under no larger group.

If the G_{2520} is contained not self-conjugately by a larger group, a C_3 must be contained by at least five G_{12} and $5 \cdot 4/1 \cdot 2\ G_{60}$. There must then be two other C_3, D and D', whose axes are interchanged by R, and one of which, say D', is commutative with A. Since D' cannot be commutative with B' without involving homologies of period 3, it must be commutative with B. It must then generate a G_{12} with B'. The only such C_3 are C' and the similar one to C' obtained by choosing the other value of b. The latter must then be D'. But it does not generate a G_{12} with C'. Hence no group can contain the G_{2520}.

* These relations follow readily from the relations of the G_{360} together with:

$$C'^3 = I, \quad E_4 C' E_4 = C'^{-1}, \quad E_1 C' = C' E_1, \quad E_2 C' = C' E_2, \quad (B'^{-1}C')^2 = I.$$

† E. H. Moore, *l. c.*; Dickson, *l. c.*

‡ The axes of C' are interchanged by this C_2 if the coefficients of the transformations are reduced modulo 7. In this case a group exists simply isomorphic with the G^7_{5040}.

§ 6. *Groups containing skew reflections which interchange each other's axes.*

Theorem 14. *The only primitive groups leaving invariant a linear complex, which do not contain skew perspectivities of higher period than 2, are of order* $16 \cdot 5$, $16 \cdot 10$, $16 \cdot 20$, 60, 120.

We consider first those groups which contain reflections whose axes are axes of a congruence belonging to the invariant complex. Any two of these reflections must leave invariant all the lines of a regulus in common and hence must be commutative, since we are assuming that no skew perspectivities of higher period than 2 are present. Moreover the product of two of these reflections must be a reflection whose axes are lines of the complex. We find that the only possible primitive groups are groups permuting five of these reflections which are all in a G_{16} (see the similar discussion under Theorem 12). The groups of this sort which contain no skew perspectivities of higher period than 2 are of order $16 \cdot 5$, $16 \cdot 10$, $16 \cdot 20$.

We suppose now that the group does not contain reflections whose axes are axes of a congruence belonging to the invariant complex. Hence the group can contain no four-group in which each reflection leaves fixed the axes of the other two, for in such a four-group the axes of one of the three reflections would be axes of a congruence. Consequently no transformation can leave fixed the axes of more than one reflection unless the axes of those reflections meet along a single line and the transformation commutative with them leaves fixed all the points on the other fixed line of the dihedral group generated by the reflections. The group leaving fixed all four vertices of a tetrahedron must be cyclic, since four of the lines of the tetrahedron must be lines of the complex and a transformation leaving fixed all the points on one of these lines must leave fixed all the points on the opposite line also. The two axes of the invariant congruence of the above group cannot be interchanged, for in that case there would be commutative transformations of the same period having different lines of fixed points, and hence the group would not be cyclic.

It will be shown later (Theorem 19) that there is no primitive group containing no commutative reflections. A reflection in the above group can be commutative with but one cyclic group having a line of fixed points, for otherwise it would be commutative with reflections which left fixed its axes. Hence all reflections commutative with any reflection in this group must leave invariant the two axes of the congruence. We shall then have three sets of reflections, those in two of the sets leaving fixed one pair of points and interchanging the other, and those in the third set interchanging both pairs of points.

The first two sets of reflections cannot be conjugate, since they are commutative with transformations of different periods. Consider a particular

reflection of the first set. The product of this reflection and any reflection conjugate with those of the second set must be of even period. Hence any reflection of the latter type must be commutative with a reflection which is commutative with this particular reflection. But no more reflections can be commutative with a reflection in either the first or second set. Also not all the reflections in the third set are commutative with the particular reflection we are considering and hence no more reflections can be commutative with them. Hence all the reflections which are conjugate with those in the second set must be in the group leaving the congruence invariant. No primitive group is therefore possible.

If no transformation except the identity leaves fixed the axes of more than one reflection, and if the order of the group leaving both axes fixed be denoted by d, then that group will be self-conjugate under a group of order $2d$ interchanging the axes. Hence, if Ω denotes the order of the whole group, there will be $\Omega/(2d)$ conjugate reflections and $(d-1)\Omega/(2d)$ transformations each leaving fixed one pair of axes.

A cyclic group of odd order, d, will be self-conjugate under a group of order d, $2d$, or $4d$. For two of the fixed lines of the cyclic group must be the axes of a congruence contained by the complex and any transformation interchanging them must be of period 4 on the vertices of the invariant tetrahedron if no skew perspectivities of higher period than 2 appear. No transformations of a cyclic group can leave invariant the axes of a reflection unless all of them do, for otherwise there would be transformations leaving fixed the axes of more than one reflection. Also if two cyclic groups of odd order contain transformations in common, the group generated by them must contain an invariant reflection whose axes are axes of a congruence belonging to the complex. Hence there will be Ω/d, $\Omega/(2d)$, or $\Omega/(4d)$ conjugate cyclic groups of order d containing $(d-1)\Omega/d$, $(d-1)\Omega/(2d)$, or $(d-1)\Omega/(4d)$ transformations.

If we attempt to enumerate the transformations which the group must contain, we are led to the following diophantine equation:

$$\Omega = 1 + \sum_{i=1}^{r} (d_i - 1)\frac{\Omega}{f_i d_i},$$

where $f_1 = 2, f_i = 1, 2, 4$ for $i > 1$. Also if $f_i = 4$, d_i can be divisible by no factors of the form $4n - 1$.

If no $f_i = 4$, the solutions of this equation are well known.* They correspond to cyclic, dihedral, tetrahedral, octahedral, and icosahedral groups. All of these groups exist here, but the icosahedral group is the only one which is primitive.

* This equation was first discussed by JORDAN, Journal für die reine und angewandte Mathematik, vol. 84 (1878), p. 89.

If one or more f_i have the value 4, it is found that $r = 3$, $f_1 = f_2 = 2$, $f_3 = 4$, since we may suppose that d_1 is divisible by 4, and if $f_i = 4$, $d_i = 5$, 13, 17, etc.

Suppose first that $d_1 = 4$. Then $d_2 = 5, 6$. If $d_2 = 5$, $d_3 \not< 13$, which is impossible. If $d_2 = 6$, then $d_3 = 5$, $\Omega = 120$.

Next suppose that $d_1 \geqq 8$. Since $d_3 \not< 5$, we must have $d_2 < 4$. If $d_2 = 3$, we must have $d_3 = 5$, $d_1 = 12$. But this is impossible since there can be but one conjugate set of C_3. We find that $d_2 = 2$ is impossible.

The solution $r = 3, f_1 = f_2 = 2, f_3 = 4, d_1 = 4, d_2 = 6, d_3 = 5, \Omega = 120$, must represent a group which can be represented as a permutation group on the six C_5. It must therefore be isomorphic with the symmetric group on 5 letters. A group of this sort occurs as a subgroup of the G_{720} and it may be shown that there is only one type of such a group which is primitive and which leaves invariant a linear complex.

Theorem 15. *The only primitive groups which contain no homologies and no skew perspectivities of higher period than 2, but which contain a G_{16} having no operator of higher period than 2, leave invariant either that G_{16} or another G_{16} of the same type.*

Consider the G_{16} generated by $S : [-x_1, -x_2, x_3, x_4]$, $T : [x_1, -x_2, -x_3, x_4]$, $U : [x_2, -x_1, -x_4, x_3]$, $V : [-x_4, -x_3, x_2, x_1]$. Any group which contains this G_{16}, but does not leave it invariant, must contain more reflections commutative with some of the reflections in it. For consider a reflection not in the G_{16}. If its product by any one of the reflections in the G_{16} is of even period it is either commutative with a reflection in the G_{16} or is commutative with one which is. If its product by every one of the reflections in the G_{16} is of odd period, it will be conjugate with all of them. If there were no more reflections commutative with any of those in the G_{16}, the G_{16} would have to remain invariant under a group which was transitive on the 15 reflections. But in any such group the reflections in the G_{16} are commutative with other reflections.

If there is an additional reflection interchanging the axes of a reflection in the G_{16}, it will either leave fixed the axes of a reflection in the G_{16} or there will be other reflections leaving fixed the axes which it interchanges. Consider a reflection W leaving fixed the axes of S. Since VW will interchange the axes of S, $(VW)^2$ must be a reflection. Moreover $(VW)^2$ must be in the G_{16}, since it will be commutative with both S and V. We suppose $(VW)^2 = SU$. Hence W must interchange the two pairs of points, $(1i00)$ and $(1 - i00)$, $(00i1)$ and $(00 - i1)$. Since the vertices of this tetrahedron are interchanged by pairs in all three ways, it may be shown that no transformation of higher period than 4 can leave them all fixed without involving a skew perspectivity of higher period than 2. Hence $(TW)^4 = I$ and

$(TUW)^4 = I$. In order that no skew perspectivity of period 4 shall appear, W must have the same fixed points as T on one of the axes of S and the same fixed points as TU on the other. Four such reflections are possible, but if one is present all will be. One of them is $W : [x_2, x_1, x_3, -x_4]$.

If a primitive group exists, S must be conjugate with reflections which interchange its axes. For otherwise S and V could not be conjugate and hence all the reflections conjugate with S would be commutative with reflections which were commutative with V. But we find it impossible to add reflections of this sort which are not conjugate with reflections interchanging their axes. Hence the only reflections with which S can be conjugate are those leaving fixed its axes. No primitive group is therefore possible.

If S is conjugate with reflections which interchange its axes, there will be a group leaving invariant a linear complex in which the axes of S and the axes of a reflection interchanging its axes are axes of congruences. In order that no skew perspectivities of period higher than 2 shall be present, this group must be primitive and of order $16 \cdot 10$.

Consider then a group of order $16 \cdot 10$. We suppose that the invariant G_{16} is that generated by S, T, U, V, and that the group of order $16 \cdot 10$ is generated by these together with W and a reflection leaving fixed the axes of V, which we denote by

$$X: \quad \left[\frac{1-i}{2}(x_2+x_3), \quad \frac{1+i}{2}(x_1-x_4), \quad \frac{1+i}{2}(x_1+x_4), \quad \frac{1-i}{2}(-x_2+x_3)\right].$$

It follows then that no more reflections can leave fixed the axes of any one of the five reflections in this group (such as S or V) whose axes are axes of congruences belonging to the complex, since they would leave the complex invariant.

Under a larger group a reflection not in the invariant G_{16} of the group of order $16 \cdot 10$ must be commutative with more reflections. Consider then a reflection Y leaving fixed the axes of X. If it is commutative with either T or V it must be commutative with both. If it is not commutative with them $(TY)^2$ must be a reflection commutative with T and hence with V, since TY interchanges the axes of X. We may then suppose that Y has this property. One of the reflections which must then be in the group is $Y: [x_1, x_3, x_2, -x_4]$. A group of order $16 \cdot 60$ is then generated, permuting five tetrahedra, one of which is $(100i)\,(100-i)\,(01i0)\,(01-i0)$.

We may show that no group can contain the group of order $16 \cdot 60$. If this is the case more reflections must be commutative with the reflections which are not in the invariant G_{16}. A reflection Z leaving fixed the axes of Y cannot interchange the axes of T, for in that case it would have to leave fixed the axes of either V or TV. Hence it must leave invariant (0001) and $(01-10)$

and interchange (1000) and (0110), or vice versa. Consider a reflection of the first sort. The fixed points of Z on the latter line must coincide with one of the two pairs of points in which the axes of X and TX meet that line, i. e. $(i-1,1,1,0)(-i+1,1,1,0)$, or $(i+1,1,1,0)(-i-1,1,1,0)$. But S, T, W, Y, Z all leave fixed (0001) and $x_4 = 0$ and must therefore generate a G_{168} in that plane. But this is not the case unless the coefficients of the transformations are reduced modulo 3.*

Theorem 16. *The only primitive group which contains reflections which interchange each other's axes, but which does not leave a linear complex invariant, does not contain a G_{16} in which there is no operator of higher period than 2, and does not contain any skew perspectivities of higher period than 2 or any homologies is a G_{168}.*

Two reflections which interchange each other's axes and a third reflection not commutative with any one of the three in the four-group generated by them must generate a group leaving a complex invariant, i. e., the complex determined by the axes of the three generating reflections. If this group is primitive, it must be either a G_{120} or a G_{60}. If it is imprimitive, it must interchange two lines of the complex and will be either an octahedral group or a group of order 48, (2, 1) isomorphic with an octahedral group.

We will show that no group of the type we are seeking can contain either a G_{120} or a G_{60}. There can be no reflection leaving fixed both axes of a reflection which belongs to either of these groups. For either this reflection would be commutative with a four-group of the G_{120} or G_{60} containing that reflection or else the square of its product by a reflection interchanging the axes which it leaves fixed would be such a reflection. But a reflection leaving fixed the axes of all three reflections in a four-group which is invariant under a tetrahedral group must remain invariant under the tetrahedral group, for otherwise there would be either a G_{16} containing no C_4 or a skew perspectivity of higher period than 2 leaving that four-group invariant. The axes of this reflection

* The fixed points of Z on the line joining (1000) to (0110) must be $(*\lambda, 1, 1, 0)$, where $\lambda^2 - \lambda + 2 = 0$. This is true of either pair of points in case the coördinates are reduced modulo 3. In this case a group of order 20160 is generated, permuting 120 points conjugate with (0001). One of the reflections which leaves fixed the axes of W we take to be

$$Z: \quad [(1-i)(x_2+x_3),\ (1+i)x_1+x_2-x_3,\ (1+i)x_1-x_2+x_3,\ x_4].$$

This group contains two conjugate sets of subgroups of order $16 \cdot 60$. The 21 subgroups in each conjugate set arrange themselves by fives in 21 ways. For there are associated with one such group five groups not conjugate with it. The $G_{16.60}$ generated by S, T, U, V, W, X, Y contains five G_{16} not conjugate with its invariant G_{16}, one of which is generated by T, V, X, Y. Any two of these five G_{16} generate the whole group of order $16 \cdot 60$. It may readily be shown that all such arrangements of 21 letters are conjugate under the symmetric group on those letters. Hence the group of order 20160 is (1, 1) isomorphic with the group of all ternary transformations with coefficients in the $GF(2^2)$ and determinant unity. It may be shown to be invariant under a group of order 40320 which contains homologies, one of these being $[x_1, x_2, x_3, -x_4]$.

must therefore be the two fixed lines of the tetrahedral group. But these lines are lines of the complex and hence the reflection would leave the complex invariant. But this is impossible.

Since a G_{120} contains two conjugate sets of reflections, a larger group containing it must contain more reflections commutative with any reflection in at least one of the two conjugate sets. Such a reflection would leave a complex invariant in common with the dihedral group in the G_{120} which leaves invariant the reflection with which it is commutative. It must then generate with that dihedral group a group leaving a regulus invariant. But there must then be reflections which leave fixed the axes of the invariant reflection.

Consider now a G_{60}. A reflection not in the G_{60} and which is not commutative with any one of the three reflections of a four-group of the G_{60} must generate with that four-group either a G_{60} or a G_{24}. If it is a G_{60} it must contain the same G_{12} leaving the four-group invariant, for otherwise there would be skew perspectivities commutative with that four-group. But it would then leave invariant the same complex, since the G_{12} leaves invariant only one complex. If it is a G_{24}, the four-group would be invariant under a G_8. This G_8 would have to leave invariant the G_{12} and hence the complex. But this is impossible, since we can have no G_{120}.

We inquire whether any primitive group of the type we are seeking can contain a G_{48}, which is $(2,1)$ isomorphic with a G_{24}. The G_{48} contains twelve reflections which interchange the axes of the invariant reflection. In a primitive group containing the G_{48} there must be more reflections interchanging the axes of these reflections. Such a reflection will leave a complex invariant in common with a dihedral G_{12} in the G_{48} which contains the reflection with which it is commutative. It must generate with that G_{12} a group which is primitive. But we have considered all such cases.

We suppose finally that any four-group in which each reflection interchanges the axes of the other two and another reflection not commutative with any one of the three in the four-group generate a G_{24}. A reflection not in a particular G_{24} must generate with the invariant four-group of that G_{24} another G_{24}. That four-group can be left invariant only by the other three four-groups of the original G_{24}. Hence this four-group can lie in but three G_{24} other than the one under which it remains invariant, i. e., G_{24} having for invariant four-groups the other three four-groups of the original G_{24}. There must be exactly two more reflections commutative with each reflection of the original G_{24} which does not lie in the invariant four-group. There must then be $3 + 6 + 6 \cdot 2$ reflections and the group must be of order $21 \cdot 8 = 168$.

A G_{168} occurs as a subgroup of the G_{2520} and it may be shown that there is only one type of such a group which is primitive.*

* Cf. BLICHFELDT, Mathematische Annalen, vol. 60 (1905), p. 226.

§ 7. *Groups containing no skew reflections which interchange each other's axes.*

Theorem 17. *The only primitive group, which contains no homologies, no skew perspectivities of higher period than 2, and no reflections interchanging each other's axes, but which contains reflections leaving fixed each other's axes and reflections whose axes meet along a single line, is a G_{60} leaving a quadric invariant.*

It may readily be shown that two reflections whose axes meet along a single line and a third reflection leaving fixed each of the axes of one of the reflections in the group generated by the first two leave at least one quadric invariant.

Any group which can be generated by two reflections whose axes meet along a single line will be contained by a group leaving a quadric invariant, since any one of the reflections in the dihedral group will be commutative with reflections which leave fixed its axes and which do not leave the dihedral group invariant. If the product of the two reflections is of period greater than 5, no such group is possible. If the product is of period 5, the only possible group is a G_{60} leaving invariant a point and a plane and a pencil of quadrics. If the product is of period 4, the only possible group is a G_{24} leaving invariant a point and a plane and a pencil of quadrics. If the product is of period 3, the group may be of either the two previous types or a G_{60} permuting five points.

Consider a G_{60} leaving invariant a point and a plane and a pencil of quadrics. We inquire whether any group can contain such a G_{60}. There can be no more reflections transforming a C_3 or a C_5 into its inverse, since such a reflection would leave invariant one of the quadrics of the family left invariant by the G_{60}. Hence if Ω denote the order of the group containing the G_{60} it must contain $(3-1)\,\Omega/(2\cdot 3)$ transformations of period 3 and $(5-1)\,\Omega/(2\cdot 5)$ transformations of period 5. We find that there can be no more reflections commutative with a reflection in the G_{60}, since every such reflection would have to generate another G_{60} with each of the two G_{10} containing the reflection with which it is commutative. A four-group will be invariant either under a group of order 48 transitive on the vertices of its invariant tetrahedron or merely under a tetrahedral group and there will be either $(16-1)\,\Omega/48$ or $(4-1)\,\Omega/12$ transformations commutative with the four-groups. The former is evidently impossible, since the number of transformations would exceed Ω. In the latter case we must have

$$\Omega = 1 + (4-1)\frac{\Omega}{12} + (3-1)\frac{\Omega}{2\cdot 3} + (5-1)\frac{\Omega}{2\cdot 5} + \cdots.$$

No other terms representing transformations can be added and we have $\Omega = 60$. Hence no group can contain the G_{60}.

Consider now a G_{24} leaving fixed a point and a plane and a pencil of quadrics. In any larger group a reflection not in the invariant four-group must be com-

mutative with more reflections. Any such reflection must generate together with the G_8 which contains the reflection with which it is commutative another G_{24}. Such a reflection will either leave invariant one quadric of the family left invariant by the original G_{24} or will leave invariant the fixed point and plane of that G_{24}. In the latter case we must have a G_{168}.

No group can contain the G_{168}. For no more reflections can be commutative with a reflection in that group or can transform a C_3 into its inverse without leaving invariant a quadric which remains invariant under a G_{24} of the G_{168}. Also a C_7 can be self-conjugate under no larger group than a G_{21}. Hence if Ω denote the order of the whole group, we must have

$$\Omega = 1 + (4-1)\frac{\Omega}{2\cdot 4} + (3-1)\frac{\Omega}{2\cdot 3} + (7-1)\frac{\Omega}{3\cdot 7} + \cdots.$$

No more terms representing transformations can be added and we have $\Omega = 168$. Hence no group can contain the G_{168}.

Consider now a G_{60} permuting five points. We may now suppose that the product of any two reflections whose axes meet along a single line is of period 3. There can be no more reflections transforming a C_3 into its inverse. For such a reflection would have to generate another G_{60} with a G_{12} containing the C_3 and hence its product by one of the reflections in the invariant four-group of the G_{12} would be of period 3. But its axes would then meet the fixed line of the C_3 which does not intersect the line of fixed points in the same points in which that line is met by the axes of a reflection in the original G_{60}. The product of these two reflections would have to be of period 3. We would then have two commutative C_3, which is impossible.

A reflection can lie in but one four-group. For such a four-group would generate with any G_6 containing the reflection of the four-group belonging to the original G_{60} another G_{60}. But all G_{60} must be conjugate, since all four-groups must be conjugate, all tetrahedral groups containing a four-group must be conjugate, and a tetrahedral group can be contained by only one G_{60}, as a C_3 can be contained by only one G_6. The two G_{60} having the G_6 in common would then have to be conjugate under a group leaving the G_6 invariant. But this would involve an impossible group commutative with a reflection.

A C_5 can be invariant only under a G_{10}. For there can be no C_4 permuting the vertices of its invariant tetrahedron, since such a C_4 would have to leave invariant the four-group commutative with its square and would hence leave the G_{60} invariant. But the only group containing the G_{60} self-conjugately is a G_{120} which contains homologies of period 2. Also if the C_5 is invariant under an abelian group of higher order, there must be additional reflections leaving the quadric invariant.

There can be no transformation leaving invariant one reflection of a four-group and interchanging the other two, since in that case a tetrahedral group would be invariant under a G_{24}. Let d_1 denote the order of the group commutative with a four-group and let Ω denote the order of the whole group. Then we must have

$$\Omega = 1 + (d_1 - 1)\frac{\Omega}{3d_1} + (3-1)\frac{\Omega}{2\cdot 3} + (5-1)\frac{\Omega}{2\cdot 5} + \cdots.$$

We must evidently have $d_1 = 4$ and no more terms representing transformations can be added. Then $\Omega = 60$. Hence no group can contain the G_{60}.

Theorem 18. *There is no primitive group which contains no homologies, no skew perspectivities of higher period than 2, no reflections interchanging each other's axes, and no reflections whose axes meet along a single line, but which contains reflections leaving fixed each other's axes.*

It may be shown that in any group of the sort described above at least two of the reflections in a four-group must be conjugate. For any reflection will certainly be conjugate with more than one-third of the reflections in the group, the only reflections with which it may not be conjugate being those commutative with it and some of the reflections commutative with the latter.

If a reflection is conjugate with one commutative with it there will be transformations sending one to the other and leaving invariant the four-group generated by the two, since all four-groups commutative with a reflection must be conjugate. Such a transformation will either leave invariant the third reflection of the four-group or will permute the three cyclically.

We consider the first case. The transformation cannot be of period 2 or 4, since there would then be reflections whose axes met along a single line. Hence it must be of higher period than 4. But then the reflection with which it is commutative can lie in but the one four-group. If a primitive group exists, all reflections must then be conjugate. Hence all six permutations will be made on the three reflections of the four-group. But in such a case any tetrahedral group containing the four-group would be invariant under an octahedral group and hence there would be reflections whose axes met along a single line.

We now suppose that a transformation which leaves a four-group invariant and is not commutative with all three reflections permutes the three cyclically. No transformation except the identity can leave more than one C_3 invariant, since we can have no reflections whose axes intersect along a single line. Hence if Ω denotes the order of the whole group and d_1 the order of the group leaving a C_3 invariant, there must be Ω/d_1 such groups and $(d_1 - 1)\,\Omega/d_1$ transformations in them.

Suppose first that a reflection lies in only one four-group. If the order of

the group commutative with a four-group be denoted by d_2, there must be $\Omega/(3d_2)$ four-groups and $(d_2-1)\,\Omega/(3d_2)$ transformations commutative with them. We should then have

$$\Omega = 1 + (d_1 - 1)\frac{\Omega}{d_1} + (d_2 - 1)\frac{\Omega}{3d_2} + \cdots.$$

We must have $d_1 = 3$ and no more terms representing transformations can be added. Then $\Omega = 3d_2$.

Suppose now that a reflection lies in more than one four-group. If the order of the cyclic group commutative with a reflection be denoted by d_2, there must be either $\Omega/(2d_2)$ reflections and $(d_2-1)\,\Omega/(2d_2)$ transformations commutative with them or else $\Omega/(4d_2)$ reflections and $(3d_2/2-1)\,\Omega/(4d_2)$ transformations commutative with them. (In the latter case one-half the transformations interchanging the vertices of the invariant tetrahedron by pairs will be of period 4.) But either case is impossible, since the number of transformations in the group would exceed Ω.

Theorem 19. *There is no primitive group containing no homologies, no skew perspectivities of higher period than 2, and no four-groups of either type.*

No transformation other than the identity can leave more than one reflection invariant unless the axes of those reflections intersect along a single line and the transformation leaves fixed all the points on the other fixed line of the dihedral group generated by the reflections. In this case one-half of the transformations which leave a reflection invariant would leave no other reflections invariant. The abelian group of order d_1 contained by this group will be self-conjugate either under the group of order $2d_1$, or under a group of order $6d_1$. In the latter case there must be also an abelian group whose order (d_2) is divisible by 3 and which can be self-conjugate only under a group of order $2d_2$. Hence the order (Ω) of the group which contains transformations leaving invariant more than one reflection must satisfy a diophantine equation of one of the two types:

$$\Omega = 1 + \tfrac{1}{2}\,\Omega + (d_1 - 1)\frac{\Omega}{2d_1} + \cdots;$$

$$\Omega = 1 + \tfrac{1}{2}\,\Omega + (d_1 - 1)\frac{\Omega}{6d_1} + (d_2 - 1)\frac{\Omega}{2d_2} + \cdots.$$

In neither case can more terms representing transformations be added. In the first case we have $\Omega = 2d_1$. In the second case we must have $d_2 = 3$, $\Omega = 6d_1$.

In case there is no transformation except the identity leaving invariant more than one reflection, we find readily by similarly constructed equations

that either there is but one reflection, or there is a dihedral group in which each reflection is commutative with itself only, or there is a group permuting cyclically the four vertices of a tetrahedron, in which each reflection is commutative with a C_4.

In case there are no reflections at all, we find readily that there is either an abelian group, or a group of order $3d$, where d denotes the order of an abelian group.

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ON THE CHARACTER OF A TRANSFORMATION IN THE NEIGHBORHOOD OF A POINT WHERE ITS JACOBIAN VANISHES*

BY

L. S. DEDERICK

If we have a transformation of n variables

$$T: \qquad y_i = \varphi_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n),$$

where in the x domain under consideration the functions φ are single-valued, and either (A) analytic functions of complex variables, or else (B) real functions of real variables with continuous partial derivatives of the first order, the character of the transformation is well known for the neighborhood of an interior point $a = (a_1, \cdots, a_n)$ in the x domain for which $J(a_1, \cdots, a_n) \neq 0$, where J is the Jacobian of the transformation,

$$J(x_1, \cdots, x_n) \equiv \frac{D(\varphi_1, \cdots, \varphi_n)}{D(x_1, \cdots, x_n)}.$$

Thus we know that in a sufficiently small neighborhood of a the transformation T possesses a single-valued inverse, having the property (A) when T has it, and the property (B) when T has it. We also know that if the x's are connected by one or more functional relations, the values at a of the resulting ordinary or partial derivatives of the x's with regard to one another or to parameters are transformed linearly by T into similar derivatives of the y's, if we make the provision in case (B) that the φ's shall have continuous derivatives of as high an order as those to be transformed. This fact as applied to first derivatives is commonly stated in the form that in a small neighborhood of the point a the transformation T is approximately projective. When $n = 2$ or 3 it leads to the geometric theorem that the order of contact of two curves or surfaces through a is unchanged by the transformation T.†

* Presented to the Society, October 26, 1912. This paper deals with a generalization of a theorem for two real variables included in the thesis presented by the author for the doctor's degree at Harvard University in 1909. This generalization was made at the suggestion of Professor Bouton, under whose direction the thesis was written. See also articles by G. R. CLEMENTS, Bulletin of the American Mathematical Society, vol. 18 (June, 1912), p. 455, Theorem X, and by S. E. URNER, these Transactions, vol. 13 (April, 1912), p. 232.

† Any of these statements which are not made explicitly in the standard treatises may be easily verified by computation. It should be noted that the proofs ordinarily given assume that the point a is an interior point of the region considered. It is easy, however, to modify the proofs so that they hold when a is a boundary point, provided the boundary locus has no singularity at a.

If on the other hand $J(a_1, \cdots, a_n) = 0$, none of these statements is necessarily true. The point a will then be called a *singular* point of the transformation; and a transformation will be said to be *regular* in a region in which it is free from singular points. The purpose of the present paper is to indicate the general character of the transformation in the neighborhood of a singular point by showing that, in general, it is essentially similar to the transformation

$$(1) \qquad \begin{aligned} y_1 &= x_1^2, \\ y_i &= x_i \qquad (i = 2, \cdots, n), \end{aligned}$$

in the neighborhood of a point where $x_1 = 0$. By "essentially similar" here is meant, differing only in properties which are altered even by a regular transformation. That is to say, the transformation of the neighborhood of a singular point by T can be obtained, in general, as the result of three transformations, of which the first and third are regular and the second has the form (1). This statement, however, is subject to the restriction of the non-vanishing at a of at least one of the n functional determinants which may be obtained from J by replacing one of the functions φ by J itself. As a matter of notation this function is assumed to be φ_1 in the explicit statement of the theorems which follow. For brevity the notation x will be used for $(x_1, \cdots, x_n)$, $y = \varphi(x)$ for the transformation T, and so on.

THEOREM 1: *If in the transformation T the functions φ are single-valued and analytic in the neighborhood of a point a, and at this point $J = 0$ but the determinant*

$$J_1(x) \equiv \frac{D(J, \varphi_2, \cdots, \varphi_n)}{D(x_1, \cdots, x_n)} \neq 0,$$

then there is a neighborhood of a in which the tranformation T can be broken up into three successive transformations, of which the first and third are regular and analytic and the second is

$$T_1: \qquad \begin{aligned} y_1' &= x_1'^2, \\ y_i' &= x_i' \qquad (i = 2, \cdots, n). \end{aligned}$$

To prove this, consider first the transformation

$$R_1: \qquad \begin{aligned} x_1'' &= J(x), \\ x_i'' &= \varphi_i(x) \qquad (i = 2, \cdots, n). \end{aligned}$$

This is analytic near a and its Jacobian is $J_1(x)$, which does not vanish at a. There will therefore be a neighborhood D of this point in which the transformation R_1 will have a single-valued analytic inverse. Let this be

$$R_1^{-1}: \qquad x = \psi(x'').$$

Its Jacobian is $1 / J_1$. We then have

$$\varphi_1(x) = \varphi_1(\psi(x'')) = \omega(x''),$$

a single-valued analytic function of x'' throughout the x'' region which is the image of D. Hence y is connected with x'' by the transformation

$R_1^{-1} T$:

$$y_1 = \omega(x''),$$
$$y_i = x_i'' \qquad (i = 2, \cdots, n).$$

Of this transformation the Jacobian is

$$\frac{\partial \omega}{\partial x_1''} = \frac{J}{J_1} = \frac{x_1''}{J_1},$$

the second form being derived from the theorem that the Jacobian of the product of two transformations is equal to the product of their Jacobians. Therefore

$$\frac{\partial^2 \omega}{\partial x_1''^2} = \frac{J_1 - x_1'' \dfrac{\partial J_1}{\partial x_1''}}{J_1^2}.$$

When $x_1'' = 0$ this becomes $1 / J_1 \neq 0$. Hence, regarding y_1 as a function of x_1'', we may write

$$\omega(x_1'', \cdots, x_n'') - \omega(0, x_2'', \cdots, x_n'') = x_1''^2 \chi(x_1'', \cdots, x_n'') = x_1''^2 \chi(x''),$$

where χ is a single-valued analytic function which remains different from zero in some neighborhood D_1 of a'', where a'' is the x'' image of a. In D_1 consider the transformation

R_2:

$$x_1' = x_1'' \sqrt{\chi(x'')},$$
$$x_i' = x_i'', \qquad (i = 2, \cdots, n).$$

where the radical denotes *one* of the two distinct functions whose square is χ and which are not connected by any branch point in D_1. The Jacobian of R_2 is

$$\frac{\partial x_1'}{\partial x_1''} = \sqrt{\chi} + \frac{x_1''}{2\sqrt{\chi}} \frac{\partial \chi}{\partial x_1''},$$

which is different from zero when $x_1'' = 0$. The transformation R_2 is therefore a regular analytic transformation in the neighborhhood of a''. Let R denote the product of the two transformations R_1 and R_2, i. e., the result of applying them in succession. Then there is a neighborhood of a in which R is a regular analytic transformation because R_1 and R_2 are such. We shall show that R may be taken as the first of the three transformations mentioned in the theorem.

To do this it is only necessary to prove that if y' is defined by the trans-

formation T_1 then y is connected with y' by a regular transformation in the neighborhood of b', where b' is the y' image of a. Now since

$$y_1' = x_1'^2 = x_1''^2 \chi(x'') = \omega(x_1'', \cdots, x_n'') - \omega(0, x_2'', \cdots, x_n'')$$
$$= y_1 - \omega(0, y_2', \cdots, y_n'),$$

it follows that y and y' are connected by the transformation

$$S: \quad \begin{aligned} y_1 &= y_1' + \omega(0, y_2', \cdots, y_n'), \\ y_i &= y_i' \qquad (i = 2, \cdots, n). \end{aligned}$$

In the neighborhood of b' this is clearly single-valued and has a Jacobian equal to unity. We have therefore effected the resolution

$$T = RT_1 S,$$

where R, T_1, and S fulfil the requirements of the theorem.

THEOREM 2: *If in the transformation T the functions φ are real and single-valued and have continuous partial derivatives of the fourth order in the neighborhood of the point a, and at this point $J = 0$ but $J_1 \neq 0$, then there is a neighborhood of a in which the transformation T can be broken up into three successive transformations of which the first and third are regular and the second is T_1.*

The proof of this is the same as that of Theorem 1, except for a few changes, of which all are obvious except perhaps one change in the transformation R_2. Here if $\chi(a'')$ is negative we must set $x_1' = x_1'' \sqrt{-\chi(x'')}$ in order that x_1' may be real.

From these two theorems we may make several inferences about the character of the transformation T at any singular point where $J_1 \neq 0$. In the first place if a is such a singular point and $b = \varphi(a)$ is its image, then all the points in the neighborhood of a which are not themselves singular points are grouped in pairs such that the two points of a pair are transformed into the same point. In the case of reals the two points of a pair lie on opposite sides of the Jacobian locus, i. e., the locus $J(x) = 0$, and all points of the neighborhood are transformed into points on the same side of the image of this locus which we shall call $\bar{J}(y) = 0$.*

Moreover, in the neighborhood of b, the inverse transformation T^{-1} is in the complex case everywhere two-valued except on $\bar{J}(y) = 0$, where the two values coincide. In the real case, the inverse exists only on one side of $\bar{J}(y) = 0$ and on this locus itself, and is two-valued except on this locus. These properties hold at any singular point of T where $J_1 \neq 0$ because they

* This fact for the transformation of two real variables was stated (without proof) by G. A. BLISS at the Princeton Colloquium in September, 1909. It had also been communicated previously by Professor Bliss to the author of this paper.

hold at any singular point of T_1 and are unaltered by a regular transformation. In the same way we can infer that all the curves through a except those having one particular direction are transformed by T into curves which are tangent to the locus $\bar{J}(y) = 0$; also that the neighborhood of $J(x) = 0$ is flattened down by the transformation into a much narrower neighborhood of $\bar{J}(y) = 0$, the degree of flattening increasing as the original neighborhood becomes narrower; so that whereas the transformation of the neighborhood of an ordinary point approximates to an ordinary projective transformation, that of a singular point where $J_1 \neq 0$ approximates to a degenerate projective transformation.

Although the proof of Theorem 2 requires the existence of partial derivatives of the fourth order for the functions φ, the properties that we have just derived for the neighborhood of a general singular point hold also if we assume only the continuity of the second order partial derivatives of the functions φ, as may be derived from the following theorem.

THEOREM 3: *If in the transformation T the functions φ are real and single-valued and have continuous partial derivatives of the second order in the neighborhood of the point a, and at this point $J = 0$ but $J_1 \neq 0$, then there is a neighborhood of a in which the transformation T can be resolved into three successive transformations, of which the first is regular, the second is T_1, and the third has a non-vanishing Jacobian but is in general two-valued.*

Let the first transformation be

$$P: \qquad \begin{aligned} x_1' &= J(x), \\ x_i' &= \varphi_i(x) \qquad (i = 2, \cdots, n). \end{aligned}$$

This has the Jacobian J_1 which does not vanish near a. Let its inverse be

$$P^{-1}: \qquad x = \psi(x'),$$

a regular transformation with the Jacobian $1/J_1$. Let the second transformation be

$$T_1: \qquad \begin{aligned} y_1' &= x_1'^2, \\ y_i' &= x_i' \qquad (i = 2, \cdots, n), \end{aligned}$$

of which the Jacobian is $2x_1'$. Here the Jacobian locus $x_1' = 0$ is the image of $J(x) = 0$. The inverse of this is

$$T_1^{-1}: \qquad \begin{aligned} x_1' &= \pm \sqrt{y_1'}, \\ x_i' &= y_i' \qquad (i = 2, \cdots, n), \end{aligned}$$

which is of course two-valued and has the Jacobian $1/2x_1'$, a two-valued

function of y'. The transformation from y' to y must then be $T_1^{-1} P^{-1} T$, or

$$Q: \quad \begin{aligned} y_1 &= \varphi_1(\psi(\pm\sqrt{y_1'}, y_2', \cdots, y_n')), \\ y_i &= y_i' \qquad (i = 2, \cdots, n). \end{aligned}$$

This has the Jacobian

$$\frac{1}{2x_1'} \cdot \frac{1}{J_1} \cdot J = \frac{x_1'}{2x_1' J_1} = \frac{\partial y_1}{\partial y_1'},$$

which is equal to $1 / 2J_1$ except perhaps when $y_1' = 0$. We can see, however, that y_1 is a continuous function of y_1' when $y_1' = 0$, and since its derivative approaches a limit at this point, the derivative also exists at this point and is equal to the limit.* Therefore $\partial y_1 / \partial y_1'$, which is the Jacobian of Q, is continuous and different from zero when $y_1' = 0$.

Hence in Q each sign of the radical gives a single-valued *regular* transformation up to *and including* $y_1' = 0$.† Moreover, on the locus $y_1' = 0$ these two transformations coincide as regards the values of the variables and of all the first partial derivatives. One of them gives the transformation of points in the y' domain which is derived from their correspondence with points on one side of $x_1' = 0$, the other that derived from their correspondence with points on the other side. Hence one of them is to be associated with one inverse of T_1, the other with the other. Thus any point y which corresponds to a point in the y' domain where y_1' is positive, is the image of two points in the x domain, one on each side of $J = 0$. In like manner we can make all the inferences from Theorem 3 that we did from Theorem 2.

Thus the properties that we derived for a singular point of T where $J_1 \neq 0$ hold if we know that the functions in T have continuous partial derivatives of the second order. The author has not been able, however, to find *single-valued* transformations for reducing T to T_1 without assuming the continuity of the fourth order derivatives in T.

PRINCETON UNIVERSITY,
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* DINI, *Funzioni di variabili reali*, § 75; or E. W. HOBSON, *Theory of functions of a real variable*, § 220.

† Since this is a boundary point of the region of definition for Q we need the modified proofs referred to in the second footnote of this article.